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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled ASPECTS OF CLASSICAL PARTICLE-LIKE FIELDS WITH FIXED CHARGE submitted by DOUGLAS STANLEY PHILLIPS in partial fulfilment of the requirements for the degree of Master of Science in Theoretical Physics.

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Supervisor

.....

.....

.....

Date 3 October 1977

TO MY PARENTS

FOR THEIR CONTINUAL INTEREST AND ENCOURAGEMENT

ABSTRACT

Several fundamental aspects of a unitary classical field theory of extended elementary particles are studied. The theory considered is based on a Lorentz invariant Lagrangian density, constructed from the electromagnetic four-vector potential, A_μ, and a real scalar field, f. The Lagrangian is not gauge invariant. Particle models are found for which a general class of localized solutions of the field equations have, at most, a discrete spectrum of allowed electric charge. Exact, finite, algebraic solutions are derived for some particular examples of discrete charge systems. The rest energy of these solutions is found to be finite and non-negative, while the angular momentum is zero. The problem of the division of the total angular momentum in classical field theory, into orbital and intrinsic components, is considered. For the type of Lagrangian studied, the integrals defining the orbital and spin angular momenta are found to be proportional for localized static solutions. An attempt is made to determine the stability of solutions in the models studied, but, due to the complexity of these systems, we are unable to reach any firm conclusion.

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NOTATION

Here we outline some of the conventions and notations that will be used throughout the thesis.

Define the coordinate four-vector

$$(x_\mu) \equiv (x_k, \text{ict}) \equiv (\vec{x}, \text{ict}),$$

where 'c' is the speed of light in vacuum. Except as otherwise noted, Latin indices, (i,j,k, etc.), take on the values 1,2,3; Greek indices, ($\alpha,\beta,\gamma,\delta$, etc.), take on the values 1,2,3,4. The summation convention is used, so, for example:

$$\sum_{\nu=1}^{4} T_{\mu\nu,\nu} \equiv T_{\mu k,k} + T_{\mu 4,4} \equiv T_{\mu\nu,\nu}.$$

Here, we have denoted partial derivatives with a comma:

$$T_{\mu\nu,\nu} \equiv \frac{\partial T_{\mu\nu}}{\partial x_\nu} \equiv \partial_\nu T_{\mu\nu} .$$

We will have occasion to consider integrals of the form

$$P_\mu = \int_\sigma T_{\mu\nu} \, d\sigma_\nu ,$$

where σ is a space-like surface, with a normal specifed by the time-like vector, $d\sigma_\nu$. If σ is a surface of constant time, then

$$(d\sigma_\nu) = (0,0,0,-\tfrac{i}{c}d^3x) ,$$

where $d^3x \equiv dx\, dy\, dz \equiv dx_1\, dx_2\, dx_3$. In this case, P_μ becomes

$$P_\mu = -\frac{i}{c}\int T_{\mu 4} \, d^3x .$$

Unless otherwise specified, the range of 3-dimensional integrations is all space.

Define the electromagnetic four-potential

$$A_\mu \equiv (A_k, i\phi) \equiv (\vec{A}, i\phi) ,$$

in terms of which one can define the electromagnetic field tensor

$$F_{\mu\nu} \equiv A_{\nu,\mu} - A_{\mu,\nu} .$$

Electric and magnetic fields, $\vec{E}$ and $\vec{H}$, respectively, are defined in the usual way by

$$\vec{E} \equiv -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \quad ; \quad \vec{H} \equiv -\vec{\nabla}\times\vec{A} .$$

The electric charge density, ρ , and the current density, j_μ, are combined into the four-vector current density, j_μ, given by

$$j_\mu \equiv (\vec{j}, ic\rho) \equiv \frac{c}{4\pi} F_{\mu\nu,\nu} .$$

We adopt the following definition for the canonical energy-momentum tensor

$$T_{\mu\nu} \equiv \delta_{\mu\nu}\mathcal{L} - A_{\sigma,\mu}\frac{\partial \mathcal{L}}{\partial A_{\sigma,\nu}} - f_{,\mu}\frac{\partial \mathcal{L}}{\partial f_{,\nu}} ,$$

where $\mathcal{L} = \mathcal{L}(A_\mu ; f ; A_{\mu,\nu} ; f_{,\nu})$ is a Lagrangian density depending only on the vector field A_μ , the scalar field f, and their derivatives. $\mathcal{L}$ does not depend explicitly on the coordinates x_μ, so one has $T_{\mu\nu,\nu} = 0$. The conserved energy-momentum, P_μ, is given by

$$(P_\mu) \equiv (P_k, iE) \equiv (\vec{P}, iE) \equiv \left(\int_\sigma T_{\mu\nu}\, d\sigma_\nu \right) ,$$

where $\vec{P}$ is the three-momentum, E is the energy, and σ is an arbitrary

space-like surface.

For definitions of the symmetrized energy-momentum tensor and the angular momentum, the reader is referred to Chapter 5. The dimensionless variables used in the remainder of the thesis are introduced in Chapter 1.

CHAPTER 1

INTRODUCTION

In this thesis, we will study several aspects of a classical field theory of extended elementary particles. As will be elaborated in the following, the theory is of the type known as unitary. In the models studied, nonlinear field equations are derived from a Lorentz invariant Lagrangian density. The Lagrangian is not gauge invariant, but we are still able to define a conserved electric charge. We attempt, with some limited success, to choose the Lagrangian such that particle-like solutions of the field equations have only a discrete spectrum of allowed charges.

In a unitary theory, particles have no properties independent of the field; particles being, simply, special solutions of the field equations. These particle-like solutions are ones in which there are localized regions where the field has a non-constant value. Particles are not assigned coordinates which satisfy equations of motion and are not attributed with mechanical properties independent of the field. The entire energy, linear momentum, and angular momentum of a particle are found by integrating the corresponding quantity for the field over the region defining the particle.

In the sixty-five years since Mie first introduced a unitary field theory[†], many authors, including Born and Infeld (1934); N. Rosen (1939); Finkelstein, Fronsdal, and Kaus (1956); Schiff (1962,1969);

[†] Mie's theory is discussed by Pauli (1958). Note, in the body of the thesis, references will be referred to by the author and year. Complete references are given at the end of the thesis.

and G. Rosen (1965), have tried this approach to the formulation of a theory of extended elementary particles. Part of the appeal of a unitary theory is aesthetic, in that the disjoint notions of particle and field are reduced to the single concept of the field, but there are other advantages as well.

In the Maxwell theory of the electrodynamics of point particles, the particles appear as singularities of the field. The field equations do not apply at the position of the particle, leading to the requirement that the field equations must be supplemented by equations of motion for the particle. In a unitary theory, one can impose the condition that the fields be finite everywhere for the solutions that represent physical particles. The field equations are satisfied at all points in space and therefore, it may be possible that no additional equations are required to complete the description of the system.

In a non-unitary theory, in which particles are assigned coordinates which obey equations of motion, there is difficulty in describing multiple particle processes, especially those involving a change in the number of particles. In the region where extended particles overlap, the particle concept loses its meaning. A unitary theory does not have this difficulty because the concept of a particle is not essential to the mathematical manipulations.

Unitary field models developed to the present, have been primarily concerned with demonstrating the existence of particle-like solutions which have appropriate interactions with other particles at large separations. Classical theories appear to be a considerable distance from treating the problem of interactions at short range, (where one would expect to require the input of quantum ideas in any case).

Perhaps one of the major reasons for the slow development of the unitary theories, is that the field equations in such an approach are necessarily nonlinear, if an interacting system is to be described. In a linear theory, the fields due to two particles may be added to give another solution of the field equations, but, in the resulting system, the presence of one particle does not affect the other. Again, one must add equations of motion for the particles in order to get a complete description of the system. On the other hand, in a unitary theory, the interactions between particles are 'built in' by way of the nonlinearities.

We now look at some of the other features of the models that we have studied. The fields used to constuct our particle models are the electromagnetic four-vector potential A_μ, and a scalar field, f. The field f was introduced by Nash (1975) as a quantity derived from a hypothetical fluctuating electromagnetic field which is assumed to permeate all space. A randomly fluctuating background field has also been used by Boyer (1975) and Schiff (1969), among others, in the context of classical field theories. The possible origin of a background field and the properties it might have, will not be discussed here, except in relation to the boundary conditions we impose on the solutions of the field equations. In a 'vacuum', where only the background radiation is present, the value, f_0, of the field f, is assumed to be a constant. We allow for both the possibilities, that f_0 is or is not equal to zero.

The fields f and A_μ satisfy coupled nonlinear equations which are derived from a Lagrangian[†], $\mathcal{L}$, through a principle of stationary action.

[†] The quantity $L = \int \mathcal{L}\, d^3x$ is the Lagrangian, so $\mathcal{L}$ is most properly referred to as the Lagrangian density. However, for convenience, we follow the common practice of calling $\mathcal{L}$, simply, the Lagrangian.

We assume that $\mathcal{L}$ depends, explicitly, only on the fields f, A_μ, and their first-order derivatives, $f_{,\nu}$ and $A_{\mu,\nu}$, respectively. Define the action, A, by

$$A \equiv \int_V \mathcal{L}\, d^3x\, dt ,$$

where V is an arbitrary four-dimensional volume. The variational principle, $\delta A = 0$, requiring that A be an extremum with respect to variations, of the fields f and A_μ, which vanish on the boundary of the volume V, gives the Euler-Lagrange field equations

$$\partial_\rho \left(\frac{\partial \mathcal{L}}{\partial A_{\sigma,\rho}} \right) = \frac{\partial \mathcal{L}}{\partial A_\sigma} \quad ; \ (\sigma = 1,2,3,4)$$

$$\partial_\rho \left(\frac{\partial \mathcal{L}}{\partial f_{,\rho}} \right) = \frac{\partial \mathcal{L}}{\partial f} .$$

There are several advantages to using a Lagrangian based theory. First, by requiring that the Lagrangian be a scalar under Lorentz transformations, the resulting field equations are automatically in covariant form. The Lorentz invariance of the Lagrangian also leads, via Noether's theorem†, directly to the conservation laws for energy, linear momentum, and the total angular momentum. Besides providing a framework for the discussion of conservation laws, the existence of a Lagrangian can be exploited in some calculations, for example, in variational approximations. Later, the extremum properties of the (integrated) Lagrangian will be used to simplify calculations of the energy in the systems that we have studied.

Another conservation law which, on the basis of present experimental evidence, should be incorporated into any model of elementary

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† See, for example, Soper (1976) , pp 101-108 .

particles, is that of the conservation of electric charge. One way to ensure that charge is conserved is to require that the Lagrangian be invariant under a gauge transformation.† However, in the models we have studied, the Lagrangian is not gauge invariant, so another method must be used. A conserved charge is introduced through the field equations, rather than through a symmetry of the Lagrangian. The Lagrangians we consider are of the form

$$\mathcal{L} = -\frac{1}{4\pi}\left(\frac{1}{4}F_{\mu\nu}^{\,2} + \frac{1}{2}(f_{,\nu})^2\right) + \bar{\mathcal{L}}\,, \tag{1.1}$$

where the term $\bar{\mathcal{L}}$ may include nonlinear terms and terms which couple the field f to the electromagnetic field. Part of the set of equations following from the Lagrangian (1.1) is of the form

$$F_{\mu\nu,\nu} = \frac{4\pi}{c} j_\mu\,. \tag{1.2}$$

We use the equation (1.2) to <u>define</u> the electric current density four-vector $j_\mu = (\vec{j},\ ic\rho)$, where $\vec{j}$ is the three-vector current density and ρ is the charge density. From (1.2), using the antisymmetry property, $F_{\mu\nu} = -F_{\nu\mu}$, we find that the current, j_μ, satisfies the usual continuity equation

$$j_{\mu,\mu} = 0\,.$$

This implies that the electric charge, q, given by

$$q \equiv \int_\sigma j_\mu\, d\sigma_\mu = \int \rho\, d^3x\,,$$

is conserved.

It is interesting that one of the major criticisms of Mie's

† Barut (1964), pp 142-144.

original work in the area of unitary field theory, was that the theory was not gauge invariant. Born and Infeld (1934) mention that Mie's theory

> " ... breaks down because Mie's field equations have the unacceptable property, that their solutions depend on the absolute value of the potential."

Pauli (1958, p 192) gives the following comments about Mie's theory:

> " Once we have found a solution for the electrostatic potential ϕ of a material particle of the required kind, ϕ + const. will not be another solution, because the field equations of Mie's theory contain the absolute value of the potential. A material particle will therefore not be able to exist in a constant external potential field. This, to us, seems to constitute a very weighty argument against Mie's theory."

However, a constant external potential is an idealization which may not be relevant to the physical world, so this does not seem like a strong objection, to us. It is important to explore the consequences of foregoing the requirement of gauge invariance, without prejudice, until more information about this approach has been accumulated.

We now briefly outline the subjects that are discussed in this thesis. In Chapter 2, we examine the question of whether or not there exist field equations in which all the particle-like solutions have the same charge. To illustrate some of the general results, specific examples are given of field equations for which a restricted class of solutions all have the same charge. However, these solutions are demonstrated to have some unphysical properties which limit their usefulness. In Chapter 3, exact algebraic solutions are found for a few particular

field theory models. The energy of these solutions is calculated in Chapter 4, and is found to be non-negative in all cases. In Chapter 5, some general comments are made about angular momentum in classical field theories, with particular emphasis on the problem of dividing the total angular momentum into spin and orbital parts. These general comments are discussed in relation to the exact solutions mentioned earlier. The angular momentum of all the solutions is found to be zero. Finally, Chapter 6 deals briefly with the problem of the stability of solutions in classical field theory. Attempts to apply stability analysis to the simplest of the models studied in the previous chapters, met with little success.

Throughout the thesis, dimensionless variables will be used to simplify the notation. Lengths will be measured in units of a fundamental length, λ. In the case where the asymptotic value, f_0, of the field f, is non-zero, it defines a corresponding length λ_0. It would be natural to make the identification $\lambda=\lambda_0$. However, in order to allow for the possibility that $\lambda_0=0$, we leave λ unspecifed. Charge will be measured in units of $\frac{e}{q_0}$, where e is the magnitude of the electron charge and q_0 is a pure number, which can be chosen according to the needs of the particular problem at hand. For example, given any solution of the field equations, having a non-zero charge, q, the number q_0 can be chosen such that q takes on any desired value. The choice of q_0, however, cannot affect the ratio of the charges of two different solutions of the field equations. Finally, velocities are measured in units of the speed of light, c.

Denoting dimensionless quantities with a hat (^), we make the following definitions:

$$(\hat{x}_\mu) \equiv (\hat{x}_k, i\hat{t}) \equiv \frac{1}{\lambda}(x_\mu),$$

$$\hat{A}_\mu \equiv \frac{\lambda}{e} q_0 A_\mu \ ,$$

$$\hat{f} \equiv \frac{\lambda^2}{e^2} q_0^2 f \ , \text{ so, } \hat{f}_0 = \frac{\lambda^2}{e^2} q_0^2 f_0 \ ,$$

$$\hat{\mathcal{L}} \equiv \frac{\lambda^4}{e^2} q_0^2 \mathcal{L}$$

Other quantities of interest are the energy and the angular momentum, which are measured in units of $\frac{e^2}{\lambda q_0^2}$ and $\frac{e^2}{c q_0^2}$ respectively.

In the remainder of the thesis, only the dimensionless quantities: $\hat{\mathcal{L}}$, $\hat{x}_\mu$, $\hat{A}_\mu$, $\hat{f}$, $\hat{f}_0$, etc. will be used but for notational convenience, we drop the hats throughout. Equivalently, we may say that $e=q_0$, and $c= \lambda = 1$, in the notation used. Note that the fundamental constant, $\hbar$ has the value $\hbar = \frac{q_0^2}{\alpha} \simeq 137\, q_0^2$ in these units, (α being the fine-structure constant), so the common natural units where $\hbar = 1$, correspond to the choice $q_0 = \alpha^{\frac{1}{2}}$.

CHAPTER 2

FIXED CHARGE SYSTEMS

2.1 Introduction to Fixed Charge Systems

One of the most general principles in physics is that the electric charge of any particle is a simple integer multiple of some fundamental unit. Also, the total charge of any system appears to be strictly conserved. The generality of these observations is evidenced by the fact that they apply to all particles, independent of their mass, spin, and the nature of their interactions with other particles. For example, a variety of experiments† show that the difference between the electron and proton charges is consistent with zero and has an upper limit of 10^{-19} or less, times the electron charge. That this is so, despite the fact that the electron appears to be point-like and does not participate in the strong interactions, whereas the proton has structure and does interact strongly, seems quite remarkable. It is natural to try to incorporate the conservation of a discrete electric charge into our particle models.

In the systems studied here, the conservation of charge has a dynamical origin, in that it derives from the field equations rather than from the invariance of the Lagrangian under some symmetry operation. Specifically, as discussed in Chapter 1, our Lagrangian is not gauge invariant. Therefore, the usual mechanism for ensuring charge conservation is not present here. In the following, we investigate the possibility that, in addition to ensuring charge conservation, the field equations

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† Dylla and King (1973) give a summary of some of the experimental results.

may also give rise to the discrete character of this conserved charge. Dirac (1931, 1948) has proposed a theory, based on the existence of the magnetic monopole, which would account for the quantization of electric charge. However, experimental searches have not yet given any firm evidence for the existence of a magnetic charge. Thus, it is possible that the origin of a discrete electric charge lies in another direction, such as the one that we explore here.

2.2 The Class of Lagrangian Studied

Ideally, one would like to guarantee that all solutions of the field equations have charges which are integer multiples of some fundamental unit. There are an unlimited number of possible Lagrangians one could consider as candidates for generating field equations with this property. However, our search was limited to a rather specific form of Lagrangian, which retains some similarity with classical electromagnetism. Furthermore, for the most part, only spherically symmetric, time-independent solutions were considered. If all the solutions of a certain class, are restricted by the field equations to having the charge $-q$, 0, or q, where q is some fixed positive number, we will call this a fixed charge system, with respect to that class of solutions. 'q' will be called the value of the fixed charge. Both $\pm q$ are allowed because particles observed in nature are seen in oppositely charged pairs.

Except for some isolated examples, only Lagrangians of the following form are considered:

$$(2.1)\quad \mathcal{L} = -\frac{1}{4\pi}\left[\frac{1}{2}\,B(f,A_\mu)\,(f_{,\nu})^2 + C(f,A_\mu)\,u_\mu F_{\mu\nu} f_{,\nu} + \frac{1}{4}\,F_{\mu\nu}{}^2 + \tilde{G}(A_\mu,f)\right]$$

Here B, C, and $\tilde{G}$ are arbitrary differentiable scalar functions of the

real scalar field f and the four-vector potential A_μ. Note that the electromagnetic term, $F_{\mu\nu}{}^2$ has the usual sign and magnitude. u_μ is a constant four-vector. For convenience, we assume that the function, C, appearing in the Lagrangian, is defined so that $u_\mu{}^2=-1$. We will examine localized, time-independent systems, where, as indicated in Appendix 1, the three-momentum of the system vanishes, so we may speak of a particle rest frame. In such a reference frame, it is assumed that u_μ reduces to the form

$$(2.2) \qquad (\, u_\mu \,) = (\, 0,\, 0,\, 0,\, iu) \,,$$

where u is a real constant, equal to ±1. The definition of u_μ as a four-vector then uniquely determines u_μ in any reference frame related to the rest frame by a Lorentz transformation. It is seen that u_μ is proportional to the four-velocity of the rest frame of the particle.

The Lagrangian (2.1) was considered by Nash (1975) in the same context, but in the present work a more systematic notation is developed and some additional cases are studied. As did Nash, we restrict our study to the case where the vector potential, $\vec{A} = 0$. The field equations for this case have been given by Nash. He found that these equations are considerably simplified if the parameters B and C, appearing in the Lagrangian, depend only on the scalar field, f, and are related by

$$B(f) = b\Big(\, C(f) \,\Big)^2 ,$$

where 'b' is an arbitrary constant. With this form for B and C, and the previously mentioned restrictions on $\vec{A}$ and u_μ, the Lagrangian (2.1), in the time-independent case, becomes:

$$(2.3) \qquad \mathcal{L} = -\frac{1}{4\pi}\left(\frac{1}{2}\, b\, (C\vec{\nabla}f)^2 + uC\, \vec{\nabla}f\cdot\vec{\nabla}\phi - \frac{1}{2}\, (\vec{\nabla}\phi)^2 + \tilde{G}(\phi,f) \right) .$$

The corresponding field equations can be written

$$(b+1)\ \nabla^2\phi = \frac{u}{C}\frac{\partial\tilde{G}}{\partial f} - b\frac{\partial\tilde{G}}{\partial\phi}$$

(2.4)

$$(b+1)\ \left(C\,\nabla^2 f + \frac{dC}{df}\,(\vec{\nabla}f)^2 \right) = \frac{1}{C}\frac{\partial\tilde{G}}{\partial f} + u\frac{\partial\tilde{G}}{\partial\phi}\ ,$$

using $u^2 = 1$.

The appearance of the combination $C\,\vec{\nabla}f$, in the Lagrangian (2.3), suggests introducing a new field $h=h(f)$, defined by

$$(2.5) \qquad h(f) \equiv \int^{f} C(f')\,df' \ ;\ \text{so}\ \vec{\nabla}h = C\,\vec{\nabla}f\ .$$

Assuming that the relation $h=h(f)$ can be inverted to give f as a function of h, the Lagrangian (2.3) takes the form

$$(2.6) \qquad \mathcal{L} = -\frac{1}{4\pi}\left[\frac{1}{2}\,b\,(\vec{\nabla}h)^2 + u\,\vec{\nabla}h\cdot\vec{\nabla}\phi - \frac{1}{2}\,(\vec{\nabla}\phi)^2 + \tilde{G}(\,\phi,\ f(h)\,) \right] .$$

Note that the coefficients of the derivative terms in the Lagrangian (2.6) are all constants. Taking the field variables to be ϕ and h, the field equations following from this Lagrangian take the simple form:

$$(b+1)\ \nabla^2\phi = u\frac{\partial\tilde{G}}{\partial h} - b\frac{\partial\tilde{G}}{\partial\phi}$$

(2.7)

$$(b+1)\ \nabla^2 h = -\frac{\partial\tilde{G}}{\partial h} + u\frac{\partial\tilde{G}}{\partial\phi}\ .$$

If one replaces h by its definition (2.5) in terms of f, these equations are seen to be the same as those derived treating ϕ and f as the field variables. Since the problem of fixing the charge depends only on the field equations, it is not affected by the change of variable from f to h. Therefore, in principle it appears that without loss of generality, one can restrict the search for fixed charge systems to Lagrangians where the coefficients of the derivative terms are constants, as in (2.3). However, in practice there are some differences between the

cases where B and C are functions of f and when they are constants. This difference results from self-imposed restrictions on the type of function one chooses for $\tilde{G}(\phi,f)$, and on the type of solutions one seeks to the field equations. For example, on the grounds of simplicity one might restrict one's study to the cases where $\tilde{G}$ is a polynomial function of its arguments. However, if $\tilde{G}$ is a polynomial in f it is not necessary that it be one in h. This is illustrated by the case C(f)=f, for which the field $h(f)=\frac{1}{2}f^2$, from the definition (2.5). If we choose, for example, $\tilde{G}(\phi,f) = (f-u\phi)^4$, then $\tilde{G}(\phi,f(h)) = ((2h)^{\frac{1}{2}}-u\phi)^4$. Similarily, for tractability, one might require that the solutions one studies have an asymptotic, $(r\to\infty)$, expansion of some specific form. For example, in a spherically symmetric case one might assume that h has the form $h = \frac{a}{r} + \frac{b}{r^2} +\ldots$ Then, if C(f)=f, as before, $f = (2h)^{\frac{1}{2}} = (\frac{2a}{r})^{\frac{1}{2}} + \ldots$ does not have an expansion of the same form, i.e. with only integral powers of $\frac{1}{r}$.

In spite of the preceding remarks, we will subsequently consider, for the most part, only those Lagrangians in which the coefficients of the derivative terms are constants. Specifically, the Lagrangian used is

$$\mathcal{L} = -\frac{1}{4\pi}\left[\frac{1}{2}B\,(\vec{\nabla}f)^2 + uC\,\vec{\nabla}f\cdot\vec{\nabla}\phi - \frac{1}{2}(\vec{\nabla}\phi)^2 + \tilde{G}(\phi,f)\right], \tag{2.8}$$

where u, B, and C are constant and $\tilde{G}$ depends on ϕ only through the scalar combinations, ϕ^2 and $u\phi$, and is a polynomial function of its arguments.

The field equations following from the Lagrangian (2.8) are

$$\begin{aligned} -\nabla^2\phi + uC\,\nabla^2 f &= \frac{\partial\tilde{G}}{\partial\phi} \\ uC\,\nabla^2\phi + B\,\nabla^2 f &= \frac{\partial\tilde{G}}{\partial f}\ . \end{aligned} \tag{2.9}$$

If $B \neq 0$, then the Lagrangian (2.8) and the equations (2.9), can be simplified somewhat, by introducing a new field variable, $\bar{f}$, given by:

$$\bar{f} \equiv f + \frac{uC}{B}\phi \tag{2.10}$$

In terms of this variable, $\bar{f}$, the Lagrangian (2.8) becomes

$$\mathcal{L} = -\frac{1}{4\pi}\left[\frac{1}{2}B(\vec{\nabla}\bar{f})^2 - \frac{1}{2}D(\vec{\nabla}\phi)^2 + \bar{G}(\phi,\bar{f})\right], \tag{2.11}$$

where

$$D \equiv 1 + \frac{u^2C^2}{B} \tag{2.12}$$

and

$$\bar{G}(\phi,\bar{f}) \equiv G(\phi,f) \quad . \tag{2.13}$$

The reason for introducing the variable, $\bar{f}$, is that the cross term, $\vec{\nabla}\phi\cdot\vec{\nabla}\bar{f}$, does not appear in the transformed Lagrangian (2.11). Treating ϕ and $\bar{f}$ as the independent field variables, the field equations derived from the Lagrangian (2.11) are

$$\begin{aligned} \nabla^2\phi &= -\frac{1}{D}\frac{\partial\bar{G}}{\partial\phi} \\ \nabla^2\bar{f} &= \frac{1}{B}\frac{\partial\bar{G}}{\partial\bar{f}} \quad , \end{aligned} \tag{2.14}$$

assuming $B \neq 0$ and $D \neq 0$. One can easily show that the equations (2.14) are equivalent to the system (2.9). Therefore, if we find that (2.14) admits only solutions in which the charge has a particular value, then the charge is also fixed in the original system (2.9). Because the form of the equations (2.14) is simpler than that of (2.9), we prefer to use the variable $\bar{f}$, rather than f.

Recall that the constant f_0 was introduced as the value of the field f, as $r\to\infty$. From the definition (2.10), we find that $\bar{f}$ satisfies this same boundary condition at infinity: $\bar{f} \to f_0$ as $r \to \infty$. We now intro-

duce another field, g, defined as that part of $\bar{f}$, which vanishes at spatial infinity:

(2.15) $$g \equiv \bar{f} - f_0 \quad .$$

We find it convenient to express the Lagrangian in terms of the field g, rather than $\bar{f}$. Again, this transformation does not affect the field equations or the problem of fixing the charge. Because f_0 is a fixed constant, $\vec{\nabla} g = \vec{\nabla} \bar{f}$, and the Lagrangian becomes

(2.16) $$\mathcal{L} = -\frac{1}{4\pi} \left(\frac{1}{2} B \, (\vec{\nabla} g)^2 - \frac{1}{2} D \, (\vec{\nabla}\phi)^2 + G(\phi, g) \right) ,$$

where

(2.17) $$G(\phi, g) \equiv \bar{G}(\phi, \bar{f}) = \tilde{G}(\phi, f) \quad .$$

The relationship between the various forms of the Lagrangian: (2.8), (2.11), and (2.16), can be made more explicit by writing out the equalities in (2.17) in an expanded form. Previously, we assumed that $\tilde{G}$ was a polynomial in the fields ϕ and f. Consequently, we can write $\tilde{G}$ as a finite summation of the form

(2.18) $$\tilde{G}(\phi, f) \equiv \sum_{m,n} \tilde{G}_{mn} \, \phi^m \, f^n \quad .$$

The summation ranges over all non-negative integers m, n for which the constants $\tilde{G}_{mn}$ are non-zero. Because the relations (2.10) and (2.15), defining $\bar{f}$ and g, respectively, are linear, $\bar{G}$ and G are also polynomials, and are of the same degree as $\tilde{G}$. We can write

(2.19) $$\bar{G}(\phi, \bar{f}) \equiv \sum_{m,n} \bar{G}_{mn} \, \phi^m \, \bar{f}^n$$

and

(2.20) $$G(\phi, g) \equiv \sum_{m,n} G_{mn} \, \phi^m \, g^n \quad .$$

Using the definitions (2.10) and (2.15), and using the binomial theorem to expand powers of f and $\bar{f}$, one finds from (2.17), that

$$\bar{G}_{mn} = \sum_{k\geq 0} \binom{n+k}{n} (-f_0)^k \, G_{m\,n+k} \,, \tag{2.21}$$

and

$$\tilde{G}_{mn} = \sum_{k=0}^{m} \binom{n+k}{n} \left(\frac{uC}{B}\right)^k \bar{G}_{m-k\,n+k}$$

$$= \sum_{k=0}^{m} \sum_{l\geq 0} \frac{(n+k+l)!}{n!k!l!} \left(\frac{uC}{B}\right)^k (-f_0)^l \, G_{m-k\,n+k+l} \,, \tag{2.22}$$

where $\binom{n+k}{n} = \frac{(n+k)!}{n!k!}$. If a fixed charge system is described in terms of the variables ϕ and g, as we will do in the following, then (2.21) and (2.22) enable one to easily find the Lagrangian (2.8) or (2.11), describing the same system in terms of the original variables.

2.3 Fixing the Charge Through Asymptotic Expansions

We now examine the possibility that the charge can be fixed in a system derived from the Lagrangian

$$\mathcal{L} = -\frac{1}{4\pi}\left[\frac{1}{2} B \, (\vec{\nabla} g)^2 - \frac{1}{2} D \, (\vec{\nabla}\phi)^2 + G(\phi,g)\right] , \tag{2.16}$$

by appropriate choice of the constants B and D, and the polynomial

$$G(\phi,g) = \sum_{m,n} G_{mn} \, \phi^m \, g^n \,. \tag{2.20}$$

The field equations derived from the Lagrangian (2.16) are, for B, D $\neq$ 0,

$$\nabla^2\phi = -\frac{1}{D}\frac{\partial G}{\partial \phi}$$

$$\nabla^2 g = \frac{1}{B}\frac{\partial G}{\partial g} \,. \tag{2.23}$$

In order to clarify our approach to fixing the charge using the field equations (2.23), we first present a simple example of a system in which all members of a class of spherically symmetric solutions, (if they exist),

have the same charge.

For the specific example, we choose B = D = 1, and

$$G(\phi,g) = \lambda\,(g-u\phi)^2 + 2\,g^3\,(g-u\phi) \quad , \tag{2.24}$$

where λ is an arbitrary non-zero real constant and $u=\pm 1$, is the fourth component of a constant four-vector, introduced in (2.2). The field equations (2.23) become

$$\nabla^2\phi = -\frac{\partial G}{\partial \phi} = 2\,\lambda\,u\,(g-u\phi) + 2\,u\,g^3 \, . \tag{2.25}$$

$$\nabla^2 g = \frac{\partial G}{\partial g} = 2\,\lambda\,(g-u\phi) + 8\,g^3 - 6\,u\,\phi\,g^2 \, . \tag{2.26}$$

Assume that a solution to the equations (2.25) and (2.26) exists, of the form

$$\begin{aligned} \phi &= \frac{q}{r} + \frac{\beta}{r^2} + \frac{\gamma}{r^3} + \frac{\delta}{r^4} + \frac{\varepsilon}{r^5} + \ldots \\ g &= \frac{a}{r} + \frac{b}{r^2} + \frac{c}{r^3} + \frac{d}{r^4} + \frac{e}{r^5} + \ldots \quad , \end{aligned} \tag{2.27}$$

in which all the parameters appearing, are constant. Substituting (2.27) into (2.25) and equating the coefficients of like powers of r on each side, one finds the conditions

$$0 = a - uq \tag{2.28.1}$$

$$0 = b - u\beta \tag{2.28.2}$$

$$0 = 2\lambda(c-u\gamma) + 2a^3 \tag{2.28.3}$$

$$2u\beta = 2\lambda(d-u\gamma) + 6a^2b \tag{2.28.4}$$

$$6u\gamma = 2\lambda(e-u\varepsilon) + 6(a^2c+ab^2) \tag{2.28.5}$$

. . .

Similarily, substitution of (2.27) into (2.26), after some simplification gives the condition:

$$6c = 2\lambda(e-u\varepsilon) + 6(a^2c+ab^2) + 6q^2(c-u\gamma) . \tag{2.29}$$

(The first four conditions derived from (2.26), analogous to (2.28.1) through (2.28.4), do not give any additional constraints in this example, so we have not written them out.) Subtracting equation (2.28.5) from (2.29) we find

$$6(c-u\gamma) = 6q^2(c-u\gamma) . \tag{2.30}$$

But, from (2.28.3) and (2.28.1), we see that $c-u\gamma$ is non-zero for charged solutions, so that it can be cancelled from (2.30) to give

$$q^2 = 1 . \tag{2.31}$$

Therefore, in this example, if a solution of the form (2.27) exists, with non-zero charge, q, then $q = \pm 1$. We say that the field equations (2.25) and (2.26) are a fixed charge system.

The method used to find systems of this type was simply to choose a form for the function $G(\phi,g)$, containing some arbitrary parameters. We then write down the field equations and substitute power series of the form (2.27), which we hope represent actual solutions to the field equations for large r. Next, the parameters appearing in G are chosen so that combining the field equations leads to a relation which limits the charge to a fixed value, as in (2.31).

In treating a more general case, where G is an arbitrary polynomial function of ϕ and g, it is convenient to introduce a systematic notation which will simplify the writing of the field equations and the

expression of our results. Define the parameters α_{mn} and β_{mn} from the field equations, by

$$\nabla^2\phi = \sum_{m,n} \beta_{mn}\, \phi^m\, g^n$$

(2.32)

$$\nabla^2 g = \sum_{m,n} \alpha_{mn}\, \phi^m\, g^n \quad .$$

Comparing (2.32) to (2.23), one finds from (2.20) that

$$\alpha_{mn} = \frac{(n+1)}{B}\, G_{m\,n+1}$$

(2.33)

$$\beta_{mn} = -\frac{(m+1)}{D}\, G_{m+1\,n} \quad .$$

Assume there exist solutions of (2.32) of the form

$$\phi = \sum_{n=1}^{\infty} \frac{b_n}{r^n}$$

(2.34)

$$g = \sum_{n=1}^{\infty} \frac{g_n}{r^n} \quad .$$

The charge, b_1, is assumed to be non-zero throughout this chapter. Define a parameter ε by

$$\varepsilon \equiv \frac{g_1}{b_1} \quad . \tag{2.35}$$

For $m = 0,1,2 \ldots$, let

$$P_m \equiv \sum_{k=0}^{m} \varepsilon^k\, \alpha_{m-k\,k} = \alpha_{m0} + \varepsilon\, \alpha_{m-1\,1} + \ldots + \varepsilon^m\, \alpha_{0m} \quad . \tag{2.36}$$

$$S_m \equiv \frac{\partial P_m}{\partial \varepsilon} = \alpha_{m-1\,1} + 2\varepsilon\, \alpha_{m-2\,2} + \ldots + m\varepsilon^{m-1}\, \alpha_{0m} \quad . \tag{2.37}$$

Similarily,

$$Q_m \equiv \sum_{k=0}^{m} \varepsilon^k\, \beta_{m-k\,k} \tag{2.38}$$

$$T_m \equiv \frac{\partial Q_m}{\partial \varepsilon} \quad . \tag{2.39}$$

Because the field equations are derived from a Lagrangian, these parameters are not all independent. The relations (2.33) can be used to show that

$$mBP_m = -DT_m + \varepsilon BS_m \quad . \tag{2.40}$$

To gain some familiarity with these definitions, we evaluate the various parameters for the example (2.24). From the field equations (2.25)and (2.26) one can easily read off the values of α_{mn} and β_{mn}, using their definition (2.32). One finds

$$\begin{aligned} &\beta_{10} = -2\lambda \; ; \; \beta_{01} = 2u\lambda \; ; \; \beta_{03} = 2u \; ; \\ &\alpha_{10} = -2u\lambda \; ; \; \alpha_{01} = 2\lambda \; ; \; \alpha_{03} = 8 \; ; \; \alpha_{12} = -6u \quad . \end{aligned} \tag{2.41}$$

The formulae (2.36) through (2.39) then give

$$\begin{aligned} &P_1 = 2(\varepsilon-u)\lambda = uQ_1 \; ; \; S_1 = 2\lambda = uT_1 \; , \text{ (using } u^2=1) \; ; \\ &P_3 = 2\varepsilon^2(4\varepsilon-3u) \; ; \; S_3 = 12\varepsilon(2\varepsilon-u) \; ; \\ &Q_3 = 2u\varepsilon^3 \; ; \; T_3 = 6u\varepsilon^2 \quad . \end{aligned} \tag{2.42}$$

Using (2.42) one can verify that the relations (2.40) are satisfied. Perhaps the most helpful property to remember, when dealing with the parameters P_m, Q_m, S_m, and T_m, is that they are derived from the coefficients of terms, $\phi^k g^l$, in the field equations, such that the subscript $m = k+l$. Therefore, from the field equations (2.25) and (2.26), for example, since there are no terms of the form ϕ^2, ϕg, or g^2 appearing, one can immediately conclude that $P_2 = Q_2 = S_2 = T_2 = 0$.

Employing the notation which has been introduced, we now summarize

the results of our search for polynomial forms for $G(\phi,g)$ which give fixed charge systems. Some of the details of the investigation are presented in Appendix 2. There, the power series expansions (2.34) are substituted into the field equations (2.32). By equating coefficients of like powers of r on each side of the equations, conditions are obtained, analogous to (2.28) and (2.29). These conditions are given by (A2.3) and (A2.4). Next, using these conditions, it was shown that the requirement that the energy of the solutions (2.34) be finite, gives only the restriction $G_{00} = 0$, on the form of the function $G(\phi,g)$ given by (2.20). Finally, conditions are established for which the charge is fixed by the field equations to a specific value for all solutions of the assumed form (2.34).

The details of how the value of the fixed charge relates to the parameters in the Lagrangian, depends on the form of the polynomial G. However, there is one feature that is common to all the fixed charge systems that were found, namely, that

$$D = \varepsilon^2 B \tag{A2.10}$$

apparently must be satisfied if the magnitude of the charge is to have a single allowed non-zero value. Recall that B and D are the constant coefficients of the derivative terms in the Lagrangian

$$\mathcal{L} = -\frac{1}{4\pi}\left[\frac{1}{2} B\,(\vec{\nabla}g)^2 - \frac{1}{2} D\,(\vec{\nabla}\phi)^2 + G(\phi,g)\right], \tag{2.16}$$

and $\varepsilon = \frac{g_1}{b_1}$ is the ratio of the leading terms in the expansions of g and ϕ. The form of G determines the value of ε. Note that if g goes to zero faster than r^{-1} as $r \to \infty$, so that $g_1 = 0$, then for charged solutions, $\varepsilon=0$. This in turn, from (A2.10), requires $D = 0$. Since D is the coefficient

of the ' electromagnetic ' term, $(\vec{\nabla}\phi)^2$, in the Lagrangian, its vanishing would be a significant departure from our requirement that the present theory be a modification of classical electrodynamics. One might expect that the case where $D \neq 0$, so $\varepsilon \neq 0$, would be of greater relevance in describing physical particles. However, as will be demonstrated later, this type of model exhibits unphysical behaviour for the interactions of the charged particle solutions, regardless of the value of ε.

We now present the specific results and illustrate with examples.

Case IA : $T_1 \neq 0$, $Q_2 \neq 0$

If the parameter $T_1 = \beta_{01}$, is non-zero, then the following conditions apparently must be satisfied if spherically symmetric solutions of the form (2.34), with finite energy, are to exist with a non-zero charge, fixed by the field equations to a single magnitude, q. (These results are derived in Appendix 2)

$$\alpha_{00}=\beta_{00}=0 \; ; \; \alpha_{10}=\varepsilon\beta_{10} \; ; \; S_1=\varepsilon T_1 \; ; \; P_2=\varepsilon Q_2 \; ;$$

(2.43)

$$T_1P_3+T_2P_2 = S_1Q_3+S_2Q_2 \quad ,$$

where ε is given by

$$\varepsilon = -\frac{\beta_{10}}{T_1} \tag{2.44}$$

If $Q_2 \neq 0$, then the value of the fixed charge, q, is given by

$$q^2 = \frac{-2Q_2}{(T_1P_4-S_1Q_4)+(T_2P_3-S_2Q_3)+(T_3P_2-S_3Q_2)+(T_1\alpha_{02}-S_1\beta_{02})\frac{Q_2^2}{T_1^2}} \tag{A2.11}$$

An example of a system of this type, satisfying all the conditions (2.43), is the Lagrangian:

$$\mathcal{L} = -\frac{1}{4\pi}\left[\frac{1}{2}(\vec{\nabla}g)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 + G\right], \text{ i.e. } B=D=1 ,$$

(2.45)

$$\text{with } G = (\alpha(g-u\phi)+\beta g^2)^2 + \frac{2}{5q^2}g^3(g-u\phi) ,$$

where $u = \pm 1$, is the fourth component of a four-vector, given by (2.2), and α and β are arbitrary non-zero constants.

The field equations are

$$\nabla^2\phi = 2u\alpha\left(\alpha(g-u\phi)+\beta g^2\right) + \frac{2u}{5q^2}g^3$$

$$\nabla^2 g = 2\alpha^2(g-u\phi) + 6\alpha\beta g^2 - 4u\alpha\beta\phi g + 4(\beta^2+\frac{2}{5q^2})g^3 - \frac{6u}{5q^2}\phi g^2 .$$

One finds that

$$\varepsilon = \frac{g_1}{b_1} = u \; ; \quad S_1 = uT_1 = 2\alpha^2 \; ;$$

$$P_2 = uQ_2 = 2\alpha\beta \; ; \; S_2 = 8u\alpha\beta \; ; \; T_2 = 4\alpha\beta \; ; \; \alpha_{02} = 6\alpha\beta \; ; \; \beta_{02} = 2u\alpha\beta \; ;$$

(2.46)

$$P_3 = 2u(\frac{1}{5q^2} + 2\beta^2) \; ; \; S_3 = 12(\frac{1}{5q^2} + \beta^2) \; ;$$

$$Q_3 = \frac{2}{5q^2} \; ; \; T_3 = \frac{6u}{5q^2} .$$

From the relations (2.46) one can verify that the conditions (2.43) are satisfied, as is (A2.10). Evaluation of (A2.11) shows that the parameter q, appearing in the Lagrangian (2.45), is the value of the fixed charge.

As a supplementary comment to this example and the others which will be presented, we give an integral expression for the rest energy of solutions of the field equations. From Appendix 1, choosing the parameter $\gamma = 1$ in (A1.11), we find that the rest energy is

$$E = \frac{1}{4\pi}\int\left(4G - (\phi\frac{\partial G}{\partial\phi} + g\frac{\partial G}{\partial g})\right)d^3x = \frac{\alpha}{2\pi}\int(g-u\phi)\left(\alpha(g-u\phi)+\beta g^2\right)d^3x .$$

We are not able to make a definite statement about the sign of the energy

in this example, without finding exact solutions.

Case IB : $T_1 \neq 0$, $Q_2 = 0$

The previous results applied only when $Q_2 \neq 0$. When $Q_2 = 0$ it is still possible to fix the charge if the following conditions are satisfied with $Q_3 \neq 0$.

$$\alpha_{00} = \beta_{00} = P_1 = Q_1 = P_2 = Q_2 = 0$$

$$P_3 = \varepsilon Q_3 \; ; \; P_4 = \varepsilon Q_4 \; ; \; S_1 = \varepsilon T_1 \; ; \; S_2 = \varepsilon T_2 \quad , \tag{2.47}$$

where as before, ε is given by (2.44). The value of the fixed charge, q, is given by

$$q^2 = \frac{-6Q_3}{T_1(P_5 - \varepsilon Q_5) - 3Q_3^2} \quad . \tag{A2.13}$$

No fixed charge system could be found with $Q_3 = 0$.

An example of a system satisfying the conditions (2.47) is

$$\mathcal{L} = -\frac{1}{4\pi}\left[\frac{1}{2}(\vec{\nabla}g)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 + G\right] \quad ,$$

$$\text{with } G = \frac{1}{2}\left[\alpha(g-u\phi)^2 + (g^2-\phi^2)\left(\left(\frac{1}{q^2}+\beta\right)\phi^2 + \left(\frac{1}{q^2}-\beta\right)g^2\right)\right] \quad , \tag{2.48}$$

where, as before, $u=\pm 1$, is the fourth component of a constant four-vector, and α and β are arbitrary non-zero constants. As indicated by the notation, the constant q appearing in G, will turn out to be the value of the fixed charge for spherically symmetric systems, as given by (A2.13).

The field equations in this example are

$$\nabla^2\phi = u\alpha(g-u\phi) + 2\phi\left(\left(\frac{1}{q^2}+\beta\right)\phi^2 - \beta g^2\right)$$

$$\nabla^2 g = \alpha(g-u\phi) + 2g\left(\beta\phi^2 + \left(\frac{1}{q^2}-\beta\right)g^2\right) \quad . \tag{2.49}$$

From the field equations we find that

$$\varepsilon = u \quad ; \quad T_1 = uS_1 = u\alpha$$

(2.50)

$$P_3 = uQ_3 = \frac{2u}{q^2} \; ; \quad T_3 = -4u\beta \; ; \quad S_3 = \frac{6}{q^2} - 4\beta \; .$$

Remembering that $u^2=1$, one can verify that the parameters (2.50) satisfy the requirements (2.47).

In this example, one finds from (A1.11), (choosing $\gamma=1$), that the rest energy of localized solutions of the field equations (2.49) is

$$(2.51) \quad E = \frac{1}{4\pi} \int \left[4G - \left(\phi \frac{\partial G}{\partial \phi} + g \frac{\partial G}{\partial g}\right) \right] d^3x = \frac{\alpha}{4\pi} \int (g - u\phi)^2 \, d^3x \; .$$

Therefore, by restricting α to positive values, we can ensure that the the rest energy is non-negative.

Another example of the case IB, (i.e. $T_1 \neq 0$, $Q_2=0$), is provided by the polynomial $G(\phi,g)$ given by (2.24), which is the example that was used to introduce the section on fixed charges.

Case II : $T_1 = 0$

The characteristic feature of the preceding examples is the appearance of linear terms in the fields ϕ and g, on the right hand side of the field equations (2.32). When $T_1 = 0$, these terms do not appear in fixed charge systems. In these systems, the following conditions are satisfied.

$$(2.52) \quad \alpha_{00}=\alpha_{01}=\alpha_{10}= 0 =\beta_{00}=\beta_{01}=\beta_{10} \; ; \; P_2=Q_2=0 \; ; \; S_2=\varepsilon T_2 \; ; \; P_3=\varepsilon Q_3 .$$

We also assume that $T_2 \neq 0$. The derivation of (2.52) is similar to the analysis of Appendix 2 for Case I.

The parameter ε is determined from the conditions $P_2=Q_2=0$. Even though these equations are quadratic in ε, (see definitions (2.36) and

(2.38)), the two equations cannot have two distinct real roots in common, if $S_2 = \varepsilon T_2$, (as is necessary to fix the charge). This is a result of a relationship between P_2 and Q_2 arising from the fact that the field equations are derived from a Lagrangian. Therefore, ε is uniquely determined by the Lagrangian for both Case I and Case II.

The value, q, of the fixed charge, for spherically symmetric solutions of a system satisfying the conditions (2.52) is given by

$$q^2 = \frac{-2Q_3}{T_2(P_4-\varepsilon Q_4) - \frac{5}{2}Q_3^2} \quad . \tag{2.53}$$

An example of this type of fixed charge system is given by

$$\mathcal{L} = -\frac{1}{4\pi}\left[\frac{1}{2}(\vec{\nabla}g)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 + G\right],$$
$$\text{with } G = \alpha g(g-u\phi)^2 + \frac{4}{5q^2}g^3(g-u\phi) \quad , \tag{2.54}$$

where $u=\pm 1$, and α and q are arbitrary non-zero constants.

The field equations derived from the Lagrangian (2.54) are

$$\nabla^2\phi = 2u\alpha g(g-u\phi) + \frac{4u}{5q^2}g^3$$

$$\nabla^2 g = \alpha(3g^2-4u\phi g+\phi^2) + \frac{4}{5q^2}(4g^3-3u\phi g^2) \quad .$$

From these equations, one finds

$$\varepsilon = u \;\; ; \;\; S_2 = uT_2 = 2u\alpha$$
$$P_3 = uQ_3 = \frac{4u}{5q^2} \;\; ; \;\; S_3 = \frac{24}{5q^2} \;\; ; \;\; T_3 = \frac{12}{5q^2}u \quad . \tag{2.55}$$

The parameters (2.55) satisfy the relations (2.52), necessary to fix the charge, and the value, q, of the resulting fixed charge, is correctly given by (2.53).

With $\gamma = \frac{1}{3}$ in (A1.11), we find that the rest energy for this example is

$$(2.56) \qquad E = \frac{1}{6\pi}\int \left(3G - \phi\frac{\partial G}{\partial \phi} + g\frac{\partial G}{\partial g}\right) d^3x = \frac{-2}{15\pi q^2}\int g^3(g-u\phi)\, d^3x \ .$$

Alternatively, with $\gamma = 1$ in (A1.11), we find

$$(2.57) \qquad E = \frac{1}{4\pi}\int \left(4G - (\phi\frac{\partial G}{\partial \phi} + g\frac{\partial G}{\partial g})\right) d^3x = \frac{\alpha}{4\pi}\int g(g-u\phi)^2\, d^3x \ .$$

From neither (2.56), nor (2.57), can we make a definite statement about the sign of the rest energy, without looking at specific solutions of the field equations.

2.4 Forces Between Particles

All of the fixed charge systems that we have found satisfy the constraint

$$(A2.10) \qquad D = \varepsilon^2 B \ .$$

This relation has important consequences with regard to the interactions between particles. Before discussing the nature of the forces between particles, there are some preliminary remarks we would like to make about the solutions to the field equations derived from the Lagrangian (2.16). We are able to calculate these forces only for certain pairs of solutions. The reason for this restriction is discussed in the following.

Some of the systems described by a Lagrangian of the form (2.16) have the undesirable feature of depending on the sign of the charge, in that the parameter u must take on two different values to ensure that an 'antiparticle' solution exists for each particle-like solution. If ($\phi=\phi_0$, $g=g_0$, $u=u_0$) is a solution, then, in general, ($\phi=-\phi_0$, $g=g_0$) will be a solu-

tion only if $u=-u_0$. The examples given for Case IA and Case II, have this property. We note that the solutions (ϕ_0,g_0,u_0) and $(-\phi_0,g_0,-u_0)$ have the same energy; a property observed for particle, antiparticle pairs occurring in nature. This follows from the expression (A1.12) for the energy, showing that it depends only on G, which in turn depends only on ϕ^2, $u\phi$, and g, by our initial assumptions.

The change in sign of u, in the Lagrangian, in order to incorporate charges of both sign into the model, causes difficulty in discussing the interactions between oppositely charged particles. In the Case IB example (2.48), however, the problem is avoided, because we can describe charges of both sign, without changing a parameter in the Lagrangian. If $(\phi=\phi_0,\ g=g_0)$ is a solution to the system (2.49), then $(\phi=-\phi_0,\ g=-g_0)$ is also a solution with the same energy, (2.51), but having the opposite charge.

Now we proceed to calculate the force between two particles, which are described by the same Lagrangian.† Thus, for systems similar to the examples of Case IA and Case II, we consider only the interaction between identical particles, but for examples similar to that for Case IB, the analysis will also apply to particles of opposite sign.

A particle as understood here, is an extended, although localized region with finite energy, where the fields, A_μ and g, have finite, nonzero values. We suppose the size of this object to be characterized by a distance, a. In the asymptotic region, at distances r>>a, from the particle, the nonlinear terms in the field equations become small compared to the linear ones. In this region one can linearly superimpose

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† A similar calculation, but for a different field theory, is given, for example, by Rosen and Rosenstock (1952).

the fields from more than one particle. However, we have no guarantee that this linear superposition is also a solution to the field equations in the region where the nonlinear terms cannot be neglected. One might even expect multiple particle solutions not to exist in the 'fixed' charge systems we have studied. However, one must remember that the charge was fixed only for spherically symmetric, static solutions, whereas systems having more than one particle present would almost certainly be non-spherical and time dependent. Whatever the actual situation, in this section we will assume that solutions do exist, representing more than a single particle.

Suppose that there are two particles, P^+ and P^-, instantaneously at rest at the points z=d and z=-d, respectively, along the z axis. At a point $r(\rho,\varphi,z)$, specified in cylindrical coordinates, far from both particles, the fields due to P^+ and P^- are approximately

$$\phi^+ = \frac{b_1^+}{r^+} \quad ; \quad g^+ = \frac{g_1^+}{r^+}$$

(2.58)

$$\text{and} \quad \phi^- = \frac{b_1^-}{r^-} \quad ; \quad g^- = \frac{g_1^-}{r^-} \quad ,$$

where $r^\pm = | \vec{r} \mp d \hat{1}_z | >> a$, $\hat{1}_z$ being a unit vector in the z direction. The total fields at $\vec{r}$ are linear superpositions of the separate contributions of the two particles:

$$\phi(\vec{r}) = \phi^+(\vec{r}) + \phi^-(\vec{r})$$

(2.59)

$$g(\vec{r}) = g^+(\vec{r}) + g^-(\vec{r}) \quad .$$

The force, $\vec{F}$, acting on a volume, V, enclosing the particle P^+, say, is the time rate of change of the momentum, P_k, in the volume. From Appendix 1 we have:

(A1.5) $$P_k = -i \int_V T_{k4}\, d^3x \quad ,$$

where $T_{\mu\nu}$ is the canonical energy-momentum tensor. Therefore,

$$F_k \equiv \frac{dP_k}{dt} = \frac{d}{dx_4} \int_V T_{k4}\, d^3x = \int_V T_{k4,4}\, d^3x = -\int T_{kj,j}\, d^3x.$$

The last step follows from the momentum conservation law $T_{k\mu,\mu} = 0$. Converting the last integral to a surface integral by the divergence theorem gives

(2.60) $$F_k = -\int_S T_{kj}\, dS_j \quad .$$

Here dS_j is the outward normal to the surface, S, enclosing the volume, V. Taking the surface, S, to be the xy plane, joined to an infinite hemisphere enclosing the space $z > 0$, we find the only contribution to the integral comes from the xy plane, because $T_{kj}dS_j$ vanishes on the hemisphere. On the xy plane, the outward normal is in the negative z direction so $(dS_j) = (0, 0, -dx\,dy)$, giving for the z component of the force:

(2.61) $$F_z = \int_{\text{xy plane}} T_{33}\, dx\, dy = \int_0^{2\pi} \int_0^{\infty} T_{33}(z=0,\rho,\varphi)\, \rho\, d\rho\, d\varphi \quad .$$

F_x and F_y vanish from symmetry considerations. To evaluate F_z we need the expression for the canonical energy-momentum tensor. This is given by

(2.62) $$\begin{aligned} T_{ij} &\equiv \delta_{ij}\mathcal{L} - \phi_{,i}\frac{\partial \mathcal{L}}{\partial \phi_{,j}} - g_{,i}\frac{\partial \mathcal{L}}{\partial g_{,j}} \\ &= \delta_{ij}\mathcal{L} + \frac{1}{4\pi}\left(B\, g_{,i}g_{,j} - D\, \phi_{,i}\phi_{,j} \right) \quad , \end{aligned}$$

where $\mathcal{L}$ is given by (2.16).

Evaluating T_{33} from (2.62), using the fields $\phi(\vec{r})$ and $g(\vec{r})$ given by (2.59), we find from (2.60):

$$
\begin{aligned}
F_z = \frac{d^2}{4}\left[B(g_1^+ - g_1^-)^2 - D(b_1^+ - b_1^-)^2 \right] \int_0^\infty \frac{\rho\, d\rho}{(\rho^2+d^2)^3} \\
- \frac{1}{4}\left[B(g_1^+ + g_1^-)^2 - D(b_1^+ + b_1^-)^2 \right] \int_0^\infty \frac{\rho^3 d\rho}{(\rho^2+d^2)^3} \\
- \frac{1}{4\pi} \int_0^{2\pi}\int_0^\infty G(\phi,g)\big|_{z=0}\, \rho\, d\rho\, d\phi \\
= \frac{b_1^+ b_1^-}{(2d)^2}(D-\varepsilon^2 B) - \frac{1}{4\pi}\int_0^{2\pi}\int_0^\infty G(\phi,g)\big|_{z=0}\, \rho\, d\rho\, d\phi \quad ,
\end{aligned}
\tag{2.63}
$$

where, in the last line we have written $g_1^+ = \varepsilon b_1^+$ and $g_1^- = \varepsilon b_1^-$.† In the case of classical electrodynamics, D = 1, B = 0, and G = 0, so we get the usual Coulomb force law:

$$F_z = \frac{b_1^+ b_1^-}{(2d)^2} \quad .$$

However, in the case of the fixed charge systems, the condition

$$D = \varepsilon^2 B \tag{A2.10}$$

must be satisfied, so that the 'inverse square' term vanishes identically. Therefore, the expression (2.63) for the force reduces to

$$F_z = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^\infty G(\phi,g)\big|_{z=0}\, \rho\, d\rho\, d\phi \quad . \tag{2.64}$$

Previously we had required the r^{-3} term in $G(\phi(\vec{r}),g(\vec{r}))$ to vanish to ensure that the energy of the particle-like solutions is finite, (see Appendix 2). However, investigation shows that for the spherically symmetric particles we have been considering, the r^{-4} term vanishes as well. This leads to the result that the force (2.64) falls off faster than the

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† For fixed charge systems, ε is uniquely determined by parameters appearing in the Lagrangian. Therefore, ε will be the same for particles P^+ and P^-, because they are described by the same Lagrangian.

inverse square of the separation of the two particles.

Because the charged solutions to the fixed charge systems studied show no Coulomb interactions, it was felt that further investigation of such systems is not warranted at this time. No attempt was made to find exact or numerical solutions to field equations of the type shown in the examples.

2.5 Other Aspects of the Charge Fixing Problem

In the preceding sections we examined the possibility of finding a Lagrangian for which the time-independent, spherically symmetric solutions, with the vector potential, $\vec{A} = 0$, all have the same magnitude of charge. Even for this restricted class of solutions, no completely acceptable model was found. By further restricting the type of solutions for which one demands that the charge be fixed, it is possible to overcome the problem of the vanishing of the Coulomb force. However, in doing so, values may appear in the spectrum of allowed charges, that are not integer multiples of a fundamental unit. This will be illustrated shortly.

Most of the remainder of the thesis will be based on the Lagrangian

$$\mathcal{L} = -\frac{1}{4\pi}\left[\frac{1}{2}(g_{,\mu})^2 + \frac{1}{4}F_{\mu\nu}{}^2 + \frac{\delta}{2}(A_{\mu,\mu})^2 + K(g)A_{\mu,\mu} - g^2A_{\mu}{}^2 + M(g)\right], \tag{2.65}$$

where K and M are differentiable functions of g, and δ is a constant. We assume that M(g) contains only powers of $g \geq 4$.

In this section we consider only the static case, with $\vec{A} = 0$, in which case (2.65) reduces to

$$\mathcal{L} = -\frac{1}{4\pi}\left[\frac{1}{2}(\vec{\nabla}g)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 + \phi^2g^2 + M\right]. \tag{2.66}$$

The field equations following from this Lagrangian are

$$\nabla^2\phi = -2g^2\phi \tag{2.67}$$

$$\nabla^2 g = 2\phi^2 g + \frac{dM}{dg} \quad . \tag{2.68}$$

Inserting the power series

$$\phi = \sum_{n=1}^{\infty} \frac{b_n}{r^n}$$

$$g = \sum_{n=1}^{\infty} \frac{g_n}{r^n}$$

into the equation (2.67) gives the condition $g_1 = 0$. For the solutions in which $g_2 \neq 0$, the charge is fixed at the value 1 by (2.68). However, in general, the value of the fixed charge, $q = b_1$, is given by $q^2 = \frac{1}{2}n(n-1)$, where n is the smallest integer such that $g_n \neq 0$. Note that the value of the charge does not depend on the particular value of g_n, but is determined by the general form of g.[†] Nash (1975) has searched for solutions to some systems of the form (2.67) and (2.68), but none was found, except in the case n=2. If solutions also exist for $n \neq 2$, then the charges will likely not be integer multiples of a fundamental unit, q being irrational for 'most' values of n.[*]

The system (2.66) is sufficiently simple that the charge can also be examined for some non-spherical solutions. Assume that series expansions exist of the form:

[†] If one had a more fundamental theory of the origin of the scalar field g, then it is possible that the asymptotic behaviour of g is completely determined. For example, one might find that $g \to \text{constant} \cdot r^{-2}$ as $r \to \infty$. In such a case the charge would be fixed to one value, rather than to a spectrum of values corresponding to different n.

[*] The smallest value of $n > 2$ such that q is rational is $n = 9$ ($q = 6$).

$$(2.69) \qquad \phi = \sum_{n=1}^{\infty} \frac{b_n(\theta,\varphi)}{r^n} \; ; \quad g = \sum_{n=1}^{\infty} \frac{g_n(\theta,\varphi)}{r^n} \; .$$

We assume that $b_n(\theta,\varphi)$ and $g_n(\theta,\varphi)$ are regular functions of the angles θ and φ.

Denote by L^2, the angular dependent part of the Laplacian operator, ∇^2, in spherical coordinates:

$$(2.70) \qquad L^2 \equiv -\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\,\frac{\partial^2\psi}{\partial\varphi^2}\right]$$

and

$$(2.71) \qquad \nabla^2\psi \equiv \frac{1}{r}\,\frac{\partial^2(r\psi)}{\partial r^2} - \frac{L^2\psi}{r^2} \; ,$$

for any twice differentiable function $\psi(r,\theta,\varphi)$.

Using the expansions (2.69) in the field equations (2.67) and (2.68), and equating the coefficients of r^{-3} on each side, one finds the condition

$$(2.72) \qquad L^2 b_1 = 2g_1^2 b_1 \; .$$

In the spherically symmetric case, b_1 is a constant so that $L^2 b_1 = 0$. Therefore, for charged solutions, where $b_1 \neq 0$, we find from (2.72) that $g_1 = 0$, as has been indicated earlier. However, the equations do not appear to prevent b_1 from having some angular dependence. This case is more difficult to treat and may be subject to difficulties of interpretation in terms of the charged particles that are seen in nature. Therefore, in the subsequent analysis, we assume that b_1 is a constant and consequently, $g_1 = 0$. With these assumptions, equating coefficients of like powers of r on each side of the field equations gives the conditions:

$$(2.73.1) \qquad (2 - L^2)\, b_2 = 0$$

$$(2.73.2) \qquad (6 - L^2)\, b_3 = -2g_2^2 b_1$$

$$\vdots \qquad\qquad \vdots$$

and

(2.74.1) $$(2 - L^2) g_2 = 2 g_2 b_1^2$$

(2.74.2) $$(6 - L^2) g_3 = 2 g_3 b_1^2 + 4 g_2 b_1 b_2 \ .$$

Equation (2.74.1) is an eigenvalue problem for the operator L^2 of the form:

(2.75) $$L^2 g_2 = \lambda g_2 \ , \text{ with } \lambda = 2(1-b_1^2) \ .$$

There is a regular, non-zero solution for (2.75) only if the eigenvalue, λ has the value †

(2.76) $$\lambda = l(l+1) \ , \text{ with } l = 0, 1, 2, \ldots \ .$$

However, to get a real solution for the charge, b_1, we are restricted, here, to the values $l=0$ and $l=1$. Corresponding to the value $l=0$, the charge is given by (2.75) and (2.76) as $b_1 = \pm 1$, whereas $l=1$ corresponds to the neutral case $b_1 = 0$. More generally, if n is the smallest value for which $g_n \neq 0$, then

$$L^2 g_n = \left(n(n-1) - 2b_1^2 \right) g_n$$

giving the condition

(2.77) $$b_1^2 = \frac{1}{2} \left(n(n-1) - l(l+1) \right) \ ; \quad l = 0, 1, 2, \ldots, n-1 \ .$$

Note that when $l = n-2$, $(n>1)$, we get $b_1^2 = n-1$, so that the possiblility exists of finding solutions with the square of the charge equal to any non-negative integer. No non-spherical solutions of the system (2.67), (2.68) have been found (except for translations of spherically symmetric solutions) so it is not known whether solutions with these non-integral

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† See, for example, Merzbacher (1970), p 182.

values of the charge are actually realized.

In the system just considered, the charge may take on some irrational values, even when restricted to spherically symmetric solutions. We now give an example where the charge spectrum allowed for spherically symmetric solutions consists only of integer values. The Lagrangian density is

$$(2.78) \qquad \mathcal{L} = -\frac{1}{4\pi}\left[\frac{1}{2}u\phi(\vec{\nabla}g)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 + u\phi^3 g^2 + \gamma g^6\right],$$

where u is the constant fourth component of a four-vector given by (2.2) and γ is an arbitrary constant.

The field equations following from the Lagrangian (2.78) are

$$\nabla^2\phi + \frac{1}{2}u(\vec{\nabla}g)^2 = -\frac{3}{2}u\phi^2 g^2$$

$$\phi\nabla^2 g + \vec{\nabla}\phi\cdot\vec{\nabla}g = \phi^3 g + \frac{6\gamma}{u}g^5$$

One finds that, if g goes as $\frac{g_n}{r^n}$ as $r\to\infty$, with $g_n\neq 0$, then the charge b_1 is fixed by the field equations, by $b_1^2 = n^2$, $n=1,2,3\ldots$. Unfortunately, as in the preceding example, one finds that angular dependent solutions apparently allow a larger spectrum of charges. In the present example, one finds $b_1^2 = n^2 - l(l+1)$; $l=0,1,2,\ldots n-1$, if non-spherical solutions exist. In particular, when $l=n-1$, we find $b_1^2 = n$. Therefore, the charge spectrum for the Lagrangian (2.78) is the same as in the case (2.67) providing solutions exist corresponding to each n and l.

These examples show that it is not adequate to consider only spherical solutions in trying to fix the charge. Although the Lagrangians were chosen so as to generate a discrete charge spectrum in the spherically symmetric case, it was found that if non-spherical solutions exist, they are also confined to a discrete set of charges. It would be interesting

to investigate the relationship, between the two types of solution, in more detail. Another problem for future study is to determine in what manner the charge allowed for time-dependent solutions is related to the charge spectrum for the static case. It is of some intrinsic interest to find a system of field equations in which the solutions, representing single particles at rest, all have the same charge. However, to fully exploit the idea of fixing the charge dynamically, through the field equations, it would seem necessary to extend the treatment to the time-dependent case. We feel that the resulting model would then be much more likely to have relevance to the physical world of time-dependent problems involving multiple particle processes.

CHAPTER 3

EXACT SOLUTIONS TO SELECTED FIELD EQUATIONS

Introduction

The existence of exact† solutions to many of the linear differential equations of relevance to classical and quantum physics, has been very valuable to the development of these disciplines. The simple harmonic oscillator is perhaps one of the best known examples of a system for which the exact solutions enjoy widespread application. Such exact solutions often serve as prototypes to illustrate some fundamental physical or mathematical principle common to a wider variety of systems. In addition, exact solutions often find use as the starting point of an approximation scheme, used to describe a more complicated system for which the solution is not precisely known.

The situation in the case of nonlinear differential equations, however, may be somewhat different. As a class, they appear to show less regularity in their properties, compared to linear equations.* General theorems governing nonlinear systems are less well developed. One might think, therefore, that the discovery of a nonlinear system for which exact solutions can be found, would be of little value because it might not share many common features with other nonlinear systems. However, it is our view that exact solutions to nonlinear equations are important, if only to develop an appreciation for the variety, if not the

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† By exact, we mean expressible in closed form in terms of well known functions.

* For a brief discussion of some of the fundamental differences between linear and nonlinear differential equations, see Davis (1962) .

regularity, displayed by nonlinear systems. If a large body of data is collected on the properties of individual nonlinear equations, then hitherto unsuspected general features may emerge.

In this chapter we present an example of a nonlinear system of equations for which particle-like solutions have been found in the time-independent, spherically symmetric cases for which the vector potential, $\vec{A} = 0$. Attempts are then made to generalize these solutions to include cases where the vector potential is non-zero. Calculation of the energy and angular momentum of these solutions is deferred to later chapters.

We look for solutions of the field equations derived from the Lagrangian (2.65) introduced in the previous chapter. Only static solutions will be sought. In the static case the Lagrangian becomes

$$(3.1) \qquad \mathcal{L} = -\frac{1}{4\pi}\left[\frac{1}{2}(\vec{\nabla}g)^2 + \frac{1}{2}(\vec{\nabla}\times\vec{A})^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{\delta}{2}(\vec{\nabla}\cdot\vec{A})^2 + K(g)\vec{\nabla}\cdot\vec{A} + g^2(\phi^2-\vec{A}^2) + M(g)\right].$$

In our attempt to find exact solutions to the fields equations derived from the Lagrangian (3.1), we consider only the cases $\delta=0$ and $\delta=1$.[†] For convenience, we also restrict the form for the functions K(g) and M(g) to polynomials in g. Further, we assume that K(g) contains only powers of g greater than or equal to 2, and that M(g) contains no powers of g less than 4. These conditions are useful in ensuring that the energy is finite and also have a bearing on the allowed values for the charge.

The field equations derived from the Lagrangian (3.1) are:

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† When $\delta=1$, the terms $-\frac{1}{4\pi}\left(\frac{1}{4}F_{\mu\nu}{}^2 - \frac{1}{2}(A_{\mu,\mu})^2\right)$ appear in the full time-dependent Lagrangian (2.65). This combination occurs in Fermi's treatment of the quantization of free electromagnetic fields. See, for example, Nishijima (1969)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})-\delta\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = 2g^2\vec{A}+\vec{\nabla}K \tag{3.2.1}$$

$$\nabla^2\phi = -2g^2\phi \tag{3.2.2}$$

$$\nabla^2 g = 2(\phi^2-\vec{A}^2)g + \frac{dK}{dg}\vec{\nabla}\cdot\vec{A} + N(g) \quad , \tag{3.2.3}$$

where $N(g) = \frac{dM}{dg}$. From our previous assumptions about the form for M(g), N(g) can be written

$$N(g) = \sum_{m\geq 3} \beta_m g^m \quad ; \quad \left(M(g) = \sum_{m\geq 4} \frac{\beta_{m-1}}{m} g^m \right) \quad . \tag{3.3}$$

If the vector potential goes to zero faster than $\frac{1}{r}$, as $r\to\infty$, then for N of the form (3.3), the equations (3.2.2) and (3.2.3) fix the charge to the value q=1, for spherically symmetric solutions with the power series expansions (2.34). If g goes as r^{-n}, as $r\to\infty$, then

$$q = \left(\frac{1}{2}n(n-1)\right)^{\frac{1}{2}} \tag{3.4}$$

is the value of the fixed charge. As we will see in a specific example, there exist solutions for which the vector potential goes as $\frac{1}{r}$ asymptotically, in which case the charge is not fixed.

Another feature of the systems we study, is that the Lagrangian (3.1) is invariant under the transformation $\phi \to -\phi$. Therefore, if ϕ is a solution to the equations (3.2), then $-\phi$ is also a solution: static solutions always occur in oppositely charged pairs. Further, the rest energy of both solutions of such a pair, is the same, as is seen from the expression (A1.15) in Appendix 1. While the existence of a degenerate pair of solutions, with charges of both sign, is a physically important property of the Lagrangian studied, there are some related problems which arise. One difficulty is that the time-dependent generalization of the Lagrangian (3.1), given by (2.65), does not automatically admit

solutions with charges of both signs. This problem arises because of the term $K(g)\, A_{\mu,\mu}$. Unless we assume that $K(g)$ changes sign with the charge, then this term will not be invariant if we replace A_μ with $-A_\mu$. But if we assume that $K(g)$ is proportional to the charge, then we have, in effect, two different Lagrangians. As mentioned in Chapter 2, this would cause some difficulty in discussing interactions between oppositely charged particles. Another situation is one in which K contains only odd powers of g and M contains even powers. Then the Lagrangian (3.1) is invariant under the transformation $(A_\mu, g) \to (-A_\mu, -g)$. In this special case charges of both sign are allowed for a single Lagrangian. If K=0, then the Lagrangian (3.1) is invariant under $A_\mu \to -A_\mu$, so that if $(\vec{A}, \phi)$ is a solution to the associated field equations, then so is $(-\vec{A}, -\phi)$. Unfortunately, $(-\vec{A}, \phi)$ and $(\vec{A}, -\phi)$ are also solutions. Therefore, unless $\vec{A}=0$, solutions will occur in quadruplets rather than in pairs. All four solutions, related by the various changes of sign, have the same rest energy. In spite of these properties, both the cases K=0 and K≠0 will be considered.

The only solutions we have found, have the field g of the form

$$g(\vec{r}) = g(r) = \frac{c}{R^2} \quad , \tag{3.5}$$

where c is a constant,

$$R \equiv (a^2 + r^2)^{\frac{1}{2}} \quad , \tag{3.6}$$

and 'a' is another constant, which is proportional to c. Nash (1975) has studied the equations (3.2.2) and (3.2.3) in the case $\vec{A}=0$, K=0, with

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† 'c' used here has no relation to the speed of light.

$N(g) = \beta_3 g^3 + \beta_4 g^4 + \beta_5 g^5$. He obtained exact and numerical, spherically symmetric solutions. In all cases g(r) was nodeless, as is the expression (3.5). Unless otherwise specified, throughout this chapter, we assume that g has the form (3.5). Note that g goes to zero as r^{-2} as $r \to \infty$, so that when the charge, q, is fixed, its value, from (3.4) is given by $q^2 = 1$. Also with the form (3.5) for g, Teshima (1976) was able to find exact solutions in the case $\delta=K=0$, $\vec{A}=0$, with a polynomial expression for N(g). In the following, we present Teshima's result and attempt to generalize it to the case of non-spherical solutions for which the vector potential, $\vec{A}$, is non-zero.

3.2 The Case K=0, δ=0

We now discuss special cases of the equations (3.2), starting with the systems in which K and δ are both zero. In this case, the field equations, (3.2) become

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = 2g^2\vec{A} \tag{3.7.1}$$

$$\nabla^2\phi = -2g^2\phi \tag{3.7.2}$$

$$\nabla^2 g = 2(\phi^2-\vec{A}^2)g + N(g) \quad , \tag{3.7.3}$$

with N(g) given by (3.3).

We first deal with the equation (3.7.2). Substituting the expression (3.5) for g into (3.7.2) and using the Laplacian operator in spherical coordinates, as given by (2.71), one has

$$\frac{1}{r}\frac{\partial^2(r\phi)}{\partial r^2} - \frac{L^2}{r^2}\phi + \frac{2c^2}{R^4}\phi = 0 \quad , \tag{3.8}$$

with L^2 given by (2.70) and R defined by (3.6).
This equation can be solved by the method of separation of variables.

Assume that there exists a solution of the form

$$\phi(\vec{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \eta_{lm}\, \psi_l(r)\, Y_{lm}(\theta,\varphi) \quad , \tag{3.9}$$

where η_{lm} are complex constants and Y_{lm} are the well known spherical harmonics. Because the Y_{lm} satisfy the relation

$$Y_{lm} = (-1)^m\, Y^*_{l\,-m} \quad ,$$

(where * denotes complex conjugation), if we choose

$$\eta_{lm} = (-1)^m\, \eta^*_{l\,-m} \quad , \tag{3.10}$$

then the requirement that $\phi(\vec{r})$ be real means that $\psi_l(\vec{r})$ is also real. Using the fact that the spherical harmonics, Y_{lm}, are eigenfunctions of the operator L^2, corresponding to the eigenvalue $l(l+1)$,† one finds that the 'radial' function $\psi_l(r)$ satisfies the equation

$$\frac{1}{r}\frac{d^2(r\psi_l)}{dr^2} + \left(\frac{2c^2}{R^4} - \frac{l(l+1)}{r^2} \right) \psi_l = 0 \quad , \tag{3.11}$$

for l=0, 1, 2, ... To solve this equation, introduce the new independent variable ρ, and the new dependent variables χ_l.

$$\rho \equiv \frac{a^2}{R^2} = \frac{a^2}{a^2 + r^2}$$

$$\chi_l \equiv \frac{R^{2l+1}}{r^l}\, \psi_l \quad ; \; l=0,1,2, \ldots \tag{3.12}$$

In terms of the new variables, (3.11) becomes

$$\rho(\rho-1)\frac{d^2\chi_l}{d\rho^2} + \left((2l+3)\rho - \frac{1}{2}(2l+3) \right) \frac{d\chi_l}{d\rho} + \frac{(2l+3)(2l+1)a^2 - 2c^2}{4a^2}\, \chi_l = 0 \tag{3.13}$$

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† See, for example, Merzbacher (1970), p 182

(3.13) is a special case of the hypergeometric equation†:

$$\rho(\rho-1)\frac{d^2\chi}{d\rho^2} + \left[(\mu+\nu+1)\rho-\lambda \right]\frac{d\chi}{d\rho} + \mu\nu\,\chi = 0 \quad , \tag{3.14}$$

where we make the identifications

$$\lambda = 1 + \frac{3}{2}$$

$$\mu = 1 + l + \frac{1}{2}\left(1+\frac{2c^2}{a^2} \right)^{\frac{1}{2}} \tag{3.15}$$

$$\nu = 1 + l - \frac{1}{2}\left(1+\frac{2c^2}{a^2} \right)^{\frac{1}{2}} .$$

The equation (3.14) has the solution

$$\chi = P\,F(\mu,\nu;\lambda;\rho) + Q\,\rho^{1-\lambda}\,F(\mu-\lambda+1,\nu-\lambda+1;2-\lambda;\rho) \quad , \tag{3.16}$$

where $F(\mu,\nu;\lambda;\rho)$ is a hypergeometric function* and P and Q are arbitrary constants. The term in Q is not finite as $\rho\to 0$, $(r\to\infty)$, so we set Q=0. For solutions in which $\lambda-\mu-\nu \leq 0$, as is the case here, where $\lambda-\mu-\nu=-(l+\frac{1}{2})$, the term in P, in general, will diverge at $\rho=1$, (r=0). An exceptional case is that in which μ or ν is a non-positive integer, in which case $F(\mu,\nu;\lambda;\rho)$ reduces to a polynomial in ρ, and is, consequently, finite in the closed interval $0\leq\rho\leq 1$, $(0\leq r)$. Therefore, from (3.15), the quantity

$$n \equiv \frac{1}{2}\left(1+\frac{2c^2}{a^2} \right)^{\frac{1}{2}} , \tag{3.17}$$

must be a positive integer if we are to get a finite solution for χ_l. Furthermore, for a given value of n, l is restricted to the values

$$l = 0,1,2, \ldots ,n-1 \quad .$$

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† The properties of the hypergeometric function used in this section, are to be found in Erdélyi et.al. (1953)

* In the context of generalized hypergeometric functions, this is denoted ${}_2F_1(\mu,\nu;\lambda;\rho)$

Returning to the original variables, one finds that the equation (3.7.2) has the solutions:

$$\phi(\vec{r}) = \phi_n(\vec{r}) = \sum_{l=0}^{n-1} \sum_{m=-l}^{l} \eta_{lm} Y_{lm}(\theta,\varphi) \phi_{nl}(r) \tag{3.18}$$

$$g(r) = g_n(r) = \frac{c_n}{R^2} \quad , \tag{3.19}$$

where

$$\phi_{nl} \equiv \frac{r^l}{R^{2l+1}} F(l+n+1, l-n+1; l+\tfrac{3}{2}; \tfrac{a^2}{R^2}) \quad , \tag{3.20}$$

$$c_n \equiv \left(\frac{4n^2-1}{2}\right)^{\frac{1}{2}} a \quad , \tag{3.21}$$

and

$$R \equiv (a^2+r^2)^{\frac{1}{2}} \quad . \tag{3.6}$$

The constants η_{lm} and the real constant 'a', may depend on the index n, which can take on the values n=1,2,3, ... , in the preceding expressions.

For some explicit calculations, it is convenient to express the hypergeometric function appearing in the solution for ϕ, in terms of the Gegenbauer polynomials[†], C_α^β. The relation is

$$F(l+n+1, l-n+1; l+\tfrac{3}{2}; \tfrac{a^2}{a^2+r^2}) = \frac{1}{\binom{n+1}{2l+1}} C_{n-l-1}^{l+1}\left(\frac{r^2-a^2}{r^2+a^2}\right) \quad , \tag{3.22}$$

where $\binom{n+1}{2l+1} = \dfrac{(n+1)!}{(2l+1)!(n-l-1)!}$.

Next consider the other field equations (3.7.1) and (3.7.3). (3.7.1) is identically satisfied by $\vec{A}=0$. In this case (3.7.3) reduces to

$$\nabla^2 g = 2\phi^2 g + N(g) \quad . \tag{3.23}$$

Since we have assumed the spherically symmetric form (3.19) for g, (3.23)

[†] Erdélyi et.al. (1953)

is satisfied only by a spherically symmetric ϕ. Therefore, in the expression (3.18) for ϕ, only the l=0 term can be included. The amplitude of ϕ is determined by the condition that the charge, q, is fixed by $q^2=1$, as mentioned earlier. Using the fact that $F(\mu,\nu;\lambda;\rho=0) = 1$, we find

$$\phi(\vec{r}) = \phi_n(\vec{r}) = \frac{q}{R} F(n+1,1-n;\frac{3}{2};\frac{a^2}{R^2}) = \frac{q}{nR} C^1_{n-1}\left(\frac{r^2-a^2}{r^2+a^2}\right) \tag{3.24}$$

and $g(\vec{r}) = g_n(r)$ is given by (3.19).

With ϕ and g given, (3.23) serves to determine

$$N(g) \equiv \sum_{m\geq 3} \beta_m g^m \quad . \tag{3.3}$$

One finds that

$$\beta_3(n) = \frac{16}{3}\frac{(n^2-4)}{(4n^2-1)} \quad ; \; n=1,2,3, \ldots$$

$$\beta_m(n) = -\frac{1}{nc_n}\left(\frac{a^2}{c_n}\right)^{m-2} \sum_{k=0}^{m-2} \frac{(-4)^k}{k+1}\binom{2n+k}{2k+1} \quad ; \; m=4,5, \ldots, 2n \; ; \; n>2 \; . \tag{3.25}$$

All other β_m are zero.

To summarize, the system of equations (3.7) is exactly satisfied by the fields $\vec{A}$, ϕ, and g, if N(g) is given by (3.3) and (3.25), with

$$\vec{A} = 0$$

$$\phi = \frac{q}{R} F(n+1,1-n;\frac{3}{2};\frac{a^2}{R^2}) \tag{3.26}$$

$$g = \frac{c_n}{R^2} \quad ,$$

for $n=1,2,3, \ldots$, $q=\pm 1$, $R = (a^2+r^2)^{\frac{1}{2}}$, and $c_n = a\left(\frac{4n^2-1}{2}\right)^{\frac{1}{2}}$.

The solutions (3.26) are those found by Teshima (1976) that were previously mentioned. An interesting feature of these solutions is that ϕ_n has (n-1) radial nodes in the open interval $0\leq r<\infty$. Therefore, for large n, the solutions have a correspondingly large number of nodes.

This is in contrast to the more usual situation, in which the known exact solutions in a nonlinear field theory are nodeless or have only a small number of nodes.

To get a better feeling for the nature of these solutions, we write out the explicit form of ϕ and N for the smallest allowed values of n.

$n=1$ $\quad N(g) = -\frac{16}{3} g^3$

$$\phi(r) = \frac{q}{R}$$

$n=2$ $\quad N(g) = -\frac{8a^4}{c^3} g^4 \quad ; \quad c=c_2=\left(\frac{15}{2}\right)^{1/4}a$

$$\phi(r) = \frac{q}{R}\left(1 - \frac{2a^2}{R^2}\right)$$

$n=3$ $\quad N(g) = \frac{16}{3}\left(\frac{1}{7}g^3 - \frac{44}{3}\frac{a^4}{c^3}g^4 + \frac{64}{3}\frac{a^6}{c^4}g^5 - \frac{32}{3}\frac{a^8}{c^5}g^6\right) \quad ; \quad c=c_3=\left(\frac{35}{2}\right)^{1/4}a$

$$\phi(r) = \frac{q}{R}\left(1 - \frac{16}{3}\frac{a^2}{R^2} + \frac{16}{3}\frac{a^4}{R^4}\right) .$$

A general feature of these solutions is that, for $n \geq 3$, N(g) includes terms from g^3 to g^{2n}. One can also see the increase, with n, of the number of radial nodes in $\phi(r)$. The charge of all the above solutions is $q=\pm 1$.

Next we investigate the possibility of finding static solutions in which the vector potential, $\vec{A}$, is non-zero. When $\vec{A} = 0$, the angular momentum of any of the time-independent localized systems we have studied, is zero. However, when $\vec{A}$ is non-zero we may be able to find a system representing a particle with non-zero intrinsic angular momentum. This would be an important addition to our particle models.

The equation (3.7.1) for the vector potential is satisfied by any solution of the system

(3.27) $$\nabla^2\vec{A} = -2g^2\vec{A}$$

(3.28) $$\vec{\nabla}\cdot\vec{A} = 0 \quad ,$$

although (3.7.1) may have other solutions as well. The reason for looking at the pair of equations (3.27) and (3.28), is that (3.27) can be solved immediately to give the Cartesian components, A_i, of the vector potential. This follows because the structure of (3.27) is the same as equation (3.7.2), which we have already solved for the scalar potential, ϕ. Comparing (3.27) to (3.18) gives

(3.29) $$A_i(r) = \sum_{l=0}^{n-1} \sum_{m=-l}^{l} (k_i)_{lm}\, Y_{lm}(\theta,\varphi)\, \phi_{nl}(r) \quad ; \quad n=1,2,3, \ldots \quad ,$$

where $(k_i)_{lm}$ are the Cartesian components of the constant vectors $\vec{k}_{lm}$, which satisfy

(3.30) $$\vec{k}_{lm} = (-1)^m \vec{k}^*_{l\,-m} \quad ,$$

and which may depend on n.

By proper choice of the constants $\vec{k}_{lm}$, the vector potential given by (3.29) can be made to satisfy the supplementary condition (3.28),and thereby providing a solution to the equation (3.7.1). The problem of imposing the condition $\vec{\nabla}\cdot\vec{A}=0$ is treated in Appendix 3. From (A3.9) we have the result that (3.7.1) is satisfied by

$$\vec{A} = A(r,\theta)\, \vec{l}_\varphi \quad ,$$

$$A(r,\theta) = \sum_{l=1}^{n-1} \left(\kappa_l \frac{r^l}{R^{2l+1}} F(l+n+1, l-n+1; l+\tfrac{3}{2}; \tfrac{a^2}{R^2}) \right.$$
$$\left. \times \sin\theta\, F(l+\tfrac{1}{2}, \tfrac{1}{2}-\tfrac{1}{2}; 2; \sin^2\theta) \right) \quad ,$$

where $\vec{l}_\varphi$ is a unit vector in the direction of increasing φ, and κ_l are

arbitrary real constants. Using the definition (3.20) for ϕ_{nl} we can simplify this expression to

$$(3.31) \qquad A(r,\theta) = \sum_{l=1}^{n-1} \kappa_l \, \phi_{nl}(r) \sin\theta \, F(l+\tfrac{1}{2},\tfrac{1}{2}-\tfrac{1}{2};2;\sin^2\theta) \quad ; \quad \vec{A} = A \, \vec{I}_\phi \, .$$

The question now arises whether the solution of (3.7.1) for $\vec{A}$, given by (3.31), and the solution of (3.7.2) for ϕ, given by (3.18), also satisfy the remaining field equation

$$(3.7.3) \qquad \nabla^2 g = 2(\phi^2-\vec{A}^2)g + N(g) \quad .$$

Since we have assumed that $g=g(r)=\frac{c}{a^2+r^2}$, a necessary condition for (3.7.3) to be satisfied is that the angular dependence in $(\phi^2-\vec{A}^2)$ vanish. Other than the case $\vec{A}=0$, we have been unable to find any solution of the given forms (3.31) and (3.18), which have this property. There appears to be at least two reasons for this failure. One problem arises from the relative sign between the term in ϕ^2 and that in $\vec{A}^2$. The highest value of l, l=n-1, in the expression (3.18) for ϕ gives rise to terms in ϕ^2 of the form $(\cos\theta)^{2n-2}$, whereas the solution (3.31) for $\vec{A}$ gives terms of the form $\sin^2\theta(\cos\theta)^{2n-4}=(1-\cos^2\theta)(\cos\theta)^{2n-4}$, in $\vec{A}^2=A^2$. Upon calculation of $\phi^2-\vec{A}^2$ one finds that the terms in $(\cos\theta)^{2n-2}$ reinforce, rather than cancel, one another.†

Another reason why no solution could be found is that the condition $\vec{\nabla}\cdot\vec{A}=0$ apparently requires that the $\frac{1}{R}$, (l=0), term in expansions for $\vec{A}$, should vanish, whereas for charged solutions the $\frac{1}{R}$ term in ϕ does not vanish.

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† Because of the recent interest in 'Euclidean' field theory (see, for example, Jackiw (1977)) we also looked for solutions to systems in which the combination $\phi^2+\vec{A}^2$, rather than $\phi^2-\vec{A}^2$, appears. However, no exact solution could be found when K=0. A solution to a system of this type, with K≠0, will be given in the next section.

This difference in the form of ϕ and $\vec{A}$ appears to prevent cancellation of the angular dependence in $\phi^2-\vec{A}^2$.

As noted in Appendix 3, the solution (3.31) for the vector potential, is not necessarily the most general solution of (3.7.1). Other possibilities include solutions where $\vec{\nabla}\cdot\vec{A} \neq 0$ and in which g is not of the simple form $g=\frac{c}{R^2}$. In particular, one could look for solutions in which g included some angular dependence. Some variational calculations were attempted in order to investigate the possibility of a non-spherical g. However, the results were generally inconclusive and will not be presented here. One of the reasons for this failure is that we had no means at our disposal to judge the accuracy of the variational 'solutions'. The Lagrangian is extremized by the exact solutions to the field equations, but for an unrestricted class of variational trial functions, the maximum-minimum character of this extremum is unknown.

3.3 The Case $\delta=1$, $K=0$

The restrictions on finding exact solutions imposed by the condition $\vec{\nabla}\cdot\vec{A}=0$, introduced in the preceding section, can be avoided by introducing a term in $(\vec{\nabla}\cdot\vec{A})^2$ into the Lagrangian. When $\delta=1$ and $K=0$ in the Lagrangian (3.1), the field equations become

$$\nabla^2\vec{A} = -2g^2\vec{A} \tag{3.32.1}$$

$$\nabla^2\phi = -2g^2\phi \tag{3.32.2}$$

$$\nabla^2 g = 2(\phi^2-\vec{A}^2)g + N(g) \quad . \tag{3.32.3}$$

For large r, we expect solutions of the equations (3.32) for ϕ and $\vec{A}$, to approach solutions of the system

$$\nabla^2 \vec{A} = 0$$

$$\nabla^2 \phi = 0 \quad .$$

These are the Maxwell equations, in the static case, if one imposes the gauge condition $\vec{\nabla}\cdot\vec{A}=0$. However, there may exist solutions of (3.32) which do not satisfy $\vec{\nabla}\cdot\vec{A}=0$. Such a solution would not tend toward a solution of the Maxwell equations, in the limit $r\to\infty$, that is, in the region where their usefulness has been well established. However, because of the paucity of nonlinear systems for which exact solutions are known, it was thought worthwhile to investigate solutions of (3.32) for which $\vec{\nabla}\cdot\vec{A}\neq 0$, even though it is expected that the results may have no physical relevance. These comments also apply to the other systems studied for which $\delta=1$ (Section 3.5).

Equations (3.32.1) and (3.32.2) have general solutions given previously by (3.18), (3.19), and (3.29) as

$$\vec{A} = \sum_{l=0}^{n-1} \sum_{m=-1}^{l} \vec{k}_{lm} Y_{lm} \phi_{nl}$$

$$\phi = \sum_{l=0}^{n-1} \sum_{m=-1}^{l} \eta_{lm} Y_{lm} \phi_{nl}$$

$$g = \frac{c_n}{R^2} \quad ,$$

where ϕ_{nl}, c_n, and R are defined by (3.20), (3.21) and (3.6), respectively. If we choose the constants η_{lm} and $\vec{k}_{lm}$, such that

$$\phi^2 - \vec{A}^2 = \phi_{n0}^{\,2}$$

(3.33)

$$\left(\phi_{n0} = \phi_{n\,l=0} = \frac{1}{R} F\left(n+1, 1-n; \frac{3}{2}; \frac{a^2}{R^2}\right) \right) \quad ,$$

then (3.32.3) is satisfied for the same choice of N(g) as in the previous section, namely, $N(g) = \sum_{m\geq 3} \beta_m g^m$, with β_m given by (3.25).

The condition (3.33) is satisfied if

$$\eta_{lm}\,\eta_{l'm'} - \vec{k}_{lm}\cdot\vec{k}_{l'm'} = 0 \quad ;\ (l,l',m,m' \text{ not all } 0)$$

(3.34)

$$(\eta_{00})^2 - (\vec{k}_{00})^2 = 4\pi \ .$$

This is easily verified by substituting the series (3.18) and (3.29) into (3.33) and using the value of the spherical harmonic $Y_{00} = (4\pi)^{-\frac{1}{2}}$ in the one term that does not cancel on the left-hand side.

The solution we have obtained to the system (3.32) suffers from the fact that a continuum of values exists for the charge, q. From the expression (3.18) for ϕ, one obtains

(3.35) $$q = \frac{1}{(4\pi)^{\frac{1}{2}}}\,\eta_{00} \ .$$

Substituting (3.35) into (3.34) gives

(3.36) $$q^2 = 1 + \frac{1}{4\pi}\,(\vec{k}_{00})^2 \ ,$$

so that the magnitude of the charge can take on any value greater than or equal to 1. The charge is 1 only when $\vec{k}_{00} = 0$, in which case $\vec{A}=0$, as follows from the relations (3.34).

3.4 The Case $\delta=0$, $K\neq 0$

The next case considered is $\delta=0$, but with no immediate restrictions on K(g). The field equations are

(3.37.1) $$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = 2g^2\vec{A} + \vec{\nabla}K$$

(3.37.2) $$\nabla^2\phi = -2g^2\phi$$

(3.37.3) $$\nabla^2 g = 2(\phi^2-\vec{A}^2)g + N(g) + \frac{dK}{dg}\vec{\nabla}\cdot\vec{A} \ .$$

When K=0 we found solutions to (3.37.1) and (3.37.2), given by (3.18) and (3.31), which also satisfied (3.37.3) only in the case $\vec{A}=0$. It was originally hoped that the introduction of the non-zero K would lead to solutions with a non-zero vector potential, $\vec{A}$, and possibly a non-zero magnetic field, $\vec{H} = \vec{\nabla}\times\vec{A}$. However, the only solutions to the system (3.37) that we have been able to find with $K\neq 0$ are of the form $\vec{A}=A(r)\ \vec{I}_r$, where $\vec{I}_r$ is the radial unit vector. The curl of any spherically symmetric radial vector field is zero so such a solution has no magnetic field.

With $\vec{A}=A(r)\ \vec{I}_r$ and $g=\frac{c}{R^2}$, equation (3.37.1) gives immediately, the result

$$A = -\frac{1}{2g^2}\frac{dg}{dr}\frac{dK}{dg} = \frac{r}{c}\frac{dK}{dg} \quad . \tag{3.38}$$

For simplicity, assume that $K = \tau g^m$, for some constant τ and where m is an integer greater than or equal to 2. From (3.38) we then find

$$A = \frac{\tau m}{c}\, r\, g^{m-1} \quad . \tag{3.39}$$

The second field equation, (3.37.2), can be satisfied in the usual way by choosing

$$\begin{aligned} \phi &= q\ \phi_{n0} = \frac{q}{R}\ F(1+n, 1-n; \frac{3}{2}; \frac{a^2}{R^2}) \\ g &= g_n = \frac{c_n}{R^2} \quad , \end{aligned} \tag{3.40}$$

with $c_n = \left(\frac{4n-1}{2}\right)^{\frac{1}{2}} a$.

From section 3.2, recall that the functions g_n and ϕ_{n0} satisfy the equation

$$\nabla^2 g_n = 2g_n\phi_{n0}^2 + N_0(g) \quad , \tag{3.41}$$

where $N_0(g) = \sum_{m\geq 3} \beta_m g^m$, with β_m determined by (3.25). Substituting (3.41) into (3.37.3), one finds that

$$(1-q^2)\,\nabla^2 g = N(g) - q^2 N_0(g) - 2\vec{A}^2 g + \frac{dK}{dg}\vec{\nabla}\cdot\vec{A} \quad . \tag{3.42}$$

All the functions, except the unknown N(g), appearing in (3.42) can be calculated in terms of g, with the result

$$N(g) = q^2 N_0(g) + \frac{2}{c}(1-q^2)\,(cg^2 - 4a^2 g^3) - \frac{m^2\tau^2}{c^2}\left[\,(3-2m)c + 2ma^2 g\,\right] g^{2m-2} \quad . \tag{3.43}$$

Previously, we have assumed that N(g) contains only powers of g greater than or equal to 3. When $m\geq 3$, this requirement on the form of N(g) is equivalent to fixing the charge to the value $q^2=1$, because otherwise a term in g^2 would appear in the expression (3.43) for N(g). With $q^2=1$, we find from (3.43) that

$$N(g) = N_0(g) - \frac{m^2\tau^2}{c^2}\left[\,(3-2m)c + 2ma^2 g\,\right] g^{2m-2} \quad ; \; m=3,4,5,\ldots \tag{3.44}$$

The case m=2 is somewhat different. One finds from (3.43) in this case:

$$N(g) = q^2 N_0(g) + \frac{2}{c^2}\,(1-q^2+2\tau^2)\,(cg^2 - 4a^2 g^3) \quad .$$

Therefore, to eliminate the g^2 term from N(g), we need

$$q^2 = 1+2\tau^2 . \tag{3.45}$$

With the charge given by (3.45), the expression for N(g) reduces to

$$N(g) = (1+2\tau^2)\,N_0(g) \quad ; \quad m=2 \quad . \tag{3.46}$$

For a given value of the parameter τ appearing in $K=\tau g^2$, the charge will be fixed at the value given by (3.45), when m=2, so $q^2=1$ only for $\tau=0$,

in this case, but for $m \geq 3$, $q^2 = 1$ for all τ.

The particular form $K = \tau g^m$ is not necessary to find solutions. By an appropriate choice of K and N one can find a wide variety of solutions for which $\vec{A} = A(r)\,\vec{1}_r$, $\phi = q\phi_{n0}$, and $g = \frac{c_n}{R^2}$.

As noted earlier, the current interest in 'Euclidean' field theory has led us to examine systems in which the combination $\phi^2 + \vec{A}^2$, rather than $\phi^2 - \vec{A}^2$, appears in the Lagrangian. Specifically, choose $m=2$, so $K = \tau g^2$. The Lagrangian studied was

$$\mathcal{L} = -\frac{1}{4\pi}\left[\frac{1}{2}(\vec{\nabla}g)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}(\vec{\nabla}\times\vec{A})^2 + K\,\vec{\nabla}\cdot\vec{A} + (\vec{A}^2+\phi^2)g^2 + M(g)\right]$$

This leads to the field equations

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = 2g^2\vec{A} - 2\tau g\vec{\nabla}g$$

$$\nabla^2\phi = -2g^2\phi \tag{3.47}$$

$$\nabla^2 g = 2(\tau\vec{\nabla}\cdot\vec{A} + \vec{A}^2 + \phi^2) + N(g) \quad .$$

For appropriate N(g), solutions were found of the form

$$g = \frac{c_n}{R^2} = \frac{\left(\frac{4n^2-1}{2}\right)^{1/2} a}{R^2}$$

$$\phi = \frac{kr}{R^3}\,F\left(2+n, 2-n; \frac{5}{2}; \frac{a^2}{R^2}\right)\cos\theta \quad , \text{ where k is a constant.}$$

$$\vec{A} = A_r(r)\,\vec{1}_r + A_\theta(r,\theta)\,\vec{1}_\theta \quad ,$$

$$\text{where } A_r = -\frac{2\tau r}{R^2}, \text{ with } \tau = \pm\frac{1}{2^{1/2}}$$

$$\text{and } A_\theta = \pm\phi\cot\theta = \pm\frac{kr}{R^3}\,F\left(2+n, 2-n; \frac{5}{2}; \frac{a^2}{R^2}\right)\sin\theta \quad .$$

In the limit $r \to \infty$, we find $\phi \to k\,\frac{\cos\theta}{r^2}$ and $\vec{A} \to -\frac{2\tau}{r}\vec{1}_r \pm k\,\frac{\sin\theta}{r^2}\vec{1}_\theta$.

The solution is seen to be neutral, and describes a particle with equal electric and magnetic dipole moments of magnitude k. (The radial term in $\vec{A}$ does not contribute to the magnetic field and therefore does not affect the dipole character of the solution.)

No other solutions to the system (3.47) were found.

3.5 The Case $\delta=1$, $K\neq 0$

The least satisfactory results were obtained in the case where both the parameter, δ, and the function, $K(g)$, are non-zero. With $\delta=1$ the field equations become:

$$\nabla^2\vec{A} = -2g^2\vec{A} - \vec{\nabla}K \tag{3.48.1}$$

$$\nabla^2\phi = -2g^2\phi \tag{3.48.2}$$

$$\nabla^2 g = 2(\phi^2-\vec{A}^2)g + \frac{dK}{dg}\vec{\nabla}\cdot\vec{A} + N(g) \quad . \tag{3.48.3}$$

Again, throughout this section we assume

$$g = \frac{c}{R^2} \text{ , where } c = c_n = \left(\frac{4n^2-1}{2}\right)^{\frac{1}{2}} a \text{ , and}$$

$$\phi = \sum_{l=0}^{n-1} \sum_{m=-1}^{l} \eta_{lm} Y_{lm} \phi_{nl} \quad , \tag{3.49}$$

with ϕ_{nl} given by (3.20).

When $K=0$, we found in (3.29) that (3.48.1) has the solution

$$\vec{A} = \vec{A}^h = \sum_{l=0}^{n-1} \sum_{m=-1}^{l} \vec{k}_{lm} Y_{lm} \phi_{nl} \quad . \tag{3.50}$$

Here we have used a superscript 'h' to denote that (3.50) is a solution of the homogeneous equation , ($K=0$) . To find the general (regular)

solution of the inhomogeneous equation (3.48.1) we only need to find any finite solution, $\vec{A}^P$, which satisfies (3.48.1) and add it to the homogeneous solution (3.50). We look for a particular solution $\vec{A}^P$ of the form

$$\vec{A}^P(r) = A^P(r)\,\vec{I}_r \tag{3.51}$$

Substituting (3.51) into (3.48.1), one finds that $A^P(r)$ satisfies the equation

$$\frac{1}{r}\frac{d^2(rA^P)}{dr^2} + 2\left(\frac{c^2}{R^4} - \frac{1}{r^2}\right) A^P = -\frac{dK}{dr} \quad . \tag{3.52}$$

The method of variation of parameters[†] can be used to find A^P because the general solution is known to the corresponding homogeneous equation

$$\frac{1}{r}\frac{d^2(rA^P)}{dr^2} + 2\left(\frac{c^2}{R^4} - \frac{1}{r^2}\right) A^P = 0 \quad . \tag{3.53}$$

The equation (3.53) is a special case, (l=1), of the equation (3.11). From (3.16) we find that the general solution of equation (3.53) is

$$A^P = S\,\frac{r}{R^3}\,F\left(n+2, 2-n; \frac{5}{2}; \frac{a^2}{R^2}\right) + T\,rF\left(n+\tfrac{1}{2}, \tfrac{1}{2}-n; -\tfrac{1}{2}; \frac{a^2}{R^2}\right) \quad . \tag{3.54}$$

In the method of variation of parameters we look for a solution in which S and T are functions of r. Substituting (3.54) into (3.52), one finds that the latter equation is satisfied if

$$\frac{dS}{dr} = \frac{dK}{dr}\frac{\psi_2}{W} \quad \text{and} \tag{3.55}$$

$$\frac{dT}{dr} = -\frac{dK}{dr}\frac{\psi_1}{W} \quad , \tag{3.56}$$

where ψ_1, ψ_2, and W are given by

[†] Ross (1964), p 118.

$$\psi_1 = \frac{r}{R^3} F(n+2, 2-n; \frac{5}{2}; \frac{a^2}{R^2}) = \phi_{n1}$$

(3.57)
$$\psi_2 = r\, F(n+\tfrac{1}{2}, \tfrac{1}{2}-n; -\tfrac{1}{2}; \frac{a^2}{R^2})$$

$$W = \psi_1 \frac{d\psi_2}{dr} - \psi_2 \frac{d\psi_1}{dr} \quad .$$

W has the simple form†, $\frac{\omega}{r^2}$, where ω is a constant determined from the functions ψ_1 and ψ_2.

No attempt was made to integrate the equations (3.55) and (3.56) for all values of n=2,3, ... , but the case n=2 was sufficiently simple that the integrations could be done for K of the form $K=\tau g^m$, with $m \geq 2$. When n=2, one finds from (3.57), that

$$\psi_1 = \frac{r}{R^3}$$

$$\psi_2 = \frac{r}{R^3} \left(r^3 + 9a^2 r - 9\frac{a^4}{r} - \frac{a^6}{r^3} \right)$$

$$W = \frac{3}{r^2}$$

leading to

(3.58)
$$S(r) = -\frac{2m\tau c^m}{3} \int^r \frac{(r^7 + 9a^2 r^5 - 9a^4 r^3 - a^6 r)}{R^{2m+5}}\, dr$$

(3.59)
$$T(r) = \frac{2m\tau c^m}{3} \int^r \frac{r^4}{R^{2m+5}}\, dr \quad .$$

These integrals can be done in terms of elementary functions for the cases where m is a positive integer or half a positive integer, but the general expressions will not be used here.

The particular solution A^p, from (3.54) and (3.57), is given by

(3.60)
$$A^p = S(r)\, \psi_1 + T(r)\, \psi_2 \quad .$$

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† The form of W, up to a multiplicative constant, is determined by the differential equation (3.52) for ψ_1 and ψ_2. See, for example, Ross (1964), p 376.

One can show that $T(r)$ and ψ_1 go to zero as $r\to\infty$, for K of the form $K=\tau g^m$, with $m\geq 2$. However, from (3.57), as $r\to\infty$, $\psi_2\to r$ so that T must vanish as fast as $\frac{1}{r^2}$, in this limit, if A^p is to vanish asymptotically. From (3.59) we find

$$\lim_{r\to\infty} T(r) = \frac{2m\tau c^m}{3}\int_0^\infty \frac{r^4}{R^{2m+5}}\,dr \quad . \tag{3.61}$$

The choice of the lower limit of integration just corresponds to a particular integration constant. This choice ensures that $\lim_{r\to 0} T(r) = 0$, which is necessary for A^p to be finite as $r\to 0$. From (3.61) we find

$$\lim_{r\to\infty} T(r) = \frac{\tau c^m}{3|a|^{2m}}\,\frac{\Gamma(\frac{5}{2})\,\Gamma(m+1)}{\Gamma(m+\frac{5}{2})} \quad , \tag{3.62}$$

where Γ is the well known special function. (3.62) shows that $\lim_{r\to\infty} T$ does not vanish for any positive m, so that the resulting A^p will not be finite. One finds, for example, when $m=2$, that A^p, as given by (3.60), is

$$A^p(r) = \frac{2\tau}{7a^2}\,r\,\left(2+\frac{15a^2}{R^2}\right) \quad .$$

These considerations show that no acceptable solution can be found in which K has the simple form, $K=\tau g^m$. However, if

$$K = \tau g^m + \eta g^p \quad ; \text{ with } p>m\geq 2, \tag{3.63}$$

the extra parameter, η, can always be chosen such that $\lim_{r\to\infty} T(r) = 0$, and an $A^p(r)$ can be found which is finite everywhere. For example, if $p=m+1$, then

$$\eta = -\frac{a^2}{c}\,\frac{(2m+5)}{(2m+2)}\,\tau \tag{3.64}$$

is the appropriate value for η. The general expression (3.60) for A^p in such a case, would, however, be quite complicated and the explicit solution was not found.

The preceding discussion has concerned only the field equation (3.48.1). Yet to be treated is the equation (3.48.3). Only in the case n=2 was any reasonable progress made in analyzing this equation. Substituting ϕ and g from (3.49), and using $\vec{A} = \vec{A}^h + A^P \vec{I}_r$, with $\vec{A}^h$ of the form (3.50), it was found from (3.48.3) that the angular dependent terms could be cancelled for a very specific form of K(g). This K is given by (3.63) and (3.64) with $m=\frac{5}{2}$, that is,

$$K(g) = g^{\frac{5}{2}} \left(1-\frac{10\,a^2}{7\,c}\, g \right) . \tag{3.65}$$

If K was not of the special form (3.65), no solutions could be found, although the possibility that some other expression for K could lead to solutions, was not entirely removed.

Because of the appearance of the fractional powers in the expression (3.65) for K and the fact that the analysis was restricted to the case n=2, the study of the case where δ and K are both non-zero was felt to be of less general interest than that of the preceding cases. In the subsequent analysis of the energy and the angular momentum of the solutions, the case $\delta=1$ and $K\neq 0$, will be excluded.

CHAPTER 4

ENERGY OF EXACT SOLUTIONS

For the time-independent systems studied in Chapter 3 we can calculate the energy corresponding to the exact solutions that have been found. The energy, so calculated, is the rest energy or rest mass of the particle described by the solution. As indicated in Appendix 1, section A1.3, the rest energy is given by

$$E = -\int \mathcal{L}\, d^3x \quad ,$$

where $\mathcal{L}$ is given by (3.1). Alternate expressions for E can be obtained by using integral relations, satisfied by solutions of the field equations obtained by considering scale and amplitude variations of the static Lagrangian. For some systems, such as the example of case IB, (see equation (2.51)), the sign of the energy can be determined from these integral expressions for E, without knowing the form of the exact solutions. However, in the examples considered in Chapter 3, this was not possible, so the integrals have to be carried out using the explicit form of the solutions.

Our calculations of the energy will be based on the expression

(A1.15) $$E = \frac{1}{12\pi}\int \left((\vec{\nabla} g)^2 + 2(\hat{A}^2-\phi^2)g^2 \right) d^3x \quad ,$$

which was derived in Appendix 1. Note that we do not need the explicit form for K(g) or M(g) to evaluate the integral (A1.15).

The first case to be considered is $\delta = K = 0$. The solution for which the energy is to be evaluated, in this case is , from (3.26) and (3.22):

$$\vec{A} = 0$$

(4.1) $$\phi = q\phi_{n0}(r) = \frac{q}{nR} C^1_{n-1}\left(\frac{r^2-a^2}{r^2+a^2}\right)$$

$$g = g_n = \frac{c_n}{R^2} \quad ,$$

where $q^2=1$, $R=(a^2+r^2)^{\frac{1}{2}}$, and $c_n=\left(\frac{4n^2-1}{2}\right)^{\frac{1}{2}} a$; $n=1,2,3, \ldots$.

Denoting the energy of the system by $E_0(n)$, (A1.15) gives

(4.2) $$E_0(n) = \frac{1}{12\pi} \int \left[(\vec{\nabla} g_n)^2 - 2\phi^2_{n0} g^2_n \right] d^3x \quad .$$

Transforming to spherical coordinates and doing the angular integration gives

(4.3) $$E_0(n) = \frac{1}{3} \int_0^\infty \left[\left(\frac{dg_n}{dr}\right)^2 - 2\phi^2_{n0}\, g^2_n \right] r^2 dr$$

$$= \frac{2}{3} c^2_n \int_0^\infty \left[\frac{2r^2}{R^8} - \frac{1}{n^2 R^6} \left[C^1_{n-1}\left(\frac{r^2-a^2}{r^2+a^2}\right) \right]^2 \right] r^2 \, dr \quad , \text{ using (4.1)}$$

$$\equiv \frac{2}{3} c_n \left(2I_1 - \frac{1}{n^2} I_2 \right), \text{ where}$$

(4.4) $$I_1 \equiv \int_0^\infty \frac{r^4}{R^8} \, dr = \frac{\pi}{32a^2|a|} \quad , \text{ and}$$

(4.5) $$I_2 \equiv \int_0^\infty \frac{r^2}{R^6} \left[C^1_{n-1}\left(\frac{r^2-a^2}{r^2+a^2}\right) \right]^2 dr$$

$$= \frac{1}{8a^2|a|} \int_{-1}^{1} (1-u^2)^{\frac{1}{2}} \left[C^1_{n-1}(u) \right]^2 du \quad ,$$

$$\text{where } u = \frac{r^2-a^2}{r^2+a^2} \quad .$$

But the integral in (4.5) is just the normalization integral for the Gegenbauer polynomials , having the value $\frac{\pi}{2}$, (Erdélyi et. al. (1953), p 177):

(4.6) $$I_2 = \frac{\pi}{16a^2|a|}$$

Substituting (4.4) and (4.6) for I_1 and I_2, respectively, into the expression (4.3) for the energy, one has

$$E_0(n) = \frac{\pi}{24|a|}\frac{c_n^2}{a^2}\left(1-\frac{1}{n^2}\right) ,$$

or using the definition of c_n,

$$E_0(n) = \frac{\pi}{48|a|}\frac{(4n^2-1)(n^2-1)}{n^2} \quad ; \quad n=1,2,3, \ \ldots \ . \tag{4.7}$$

The rest energy is seen to be positive for n=2,3, ... , but is 0 for n=1. Because the parameter, a, can depend on n, the spectrum of rest energies is not restricted to the explicit dependence on n given in (4.7), but can take on any form, (except for n=1, for which E_0 is always 0). 'a' characterizes the size of the particle, in the sense that the value of g at r=a is one half the maximum value, which occurs at r=0. Therefore, to say that 'a' depends on n is equivalent to saying that the particles corresponding to different n may be different sizes.

The next case to consider is δ=1, K=0, for which we found solutions given by (3.18), (3.29), and (3.19) as

$$\vec{A} = \sum_{l=0}^{n-1}\sum_{m=-l}^{l} \vec{k}_{lm}\, Y_{lm}(\theta,\varphi)\, \phi_{nl}(r)$$

$$\phi = \sum_{l=0}^{n-1}\sum_{m=-l}^{l} \eta_{lm}\, Y_{lm}(\theta,\varphi)\, \phi_{nl}(r) \tag{4.8}$$

$$g = g_n = \frac{c_n}{R^2}$$

The parameters $\vec{k}_{lm}$ and η_{lm} are chosen such that

$$\phi^2 - \vec{A}^2 = \phi_{n0}^2 \tag{3.30}$$

Therefore, the energy, as given by (A1.15), is

$$E(n) = \frac{1}{12\pi}\int\left((\vec{\nabla}g)^2 + 2(\vec{A}^2-\phi^2)g^2 \right) d^3x$$

$$= \frac{1}{12\pi}\int\left((\vec{\nabla}g_n)^2 - 2\phi_{n0}^2\, g_n^2 \right) d^3x \qquad (4.9)$$

$$= E_0(n) \quad , \text{ from (4.2)},$$

where $E_0(n)$ is given by (4.7).

Therefore, the rest energy spectrum for the solutions (4.8) is the same as that of the spherically symmetric solutions (4.1). To each discrete value of the energy, however, there is an infinite number of solutions, corresponding to various values of the constants $\vec{k}_{lm}$ and η_{lm}. Even if one were to restrict solutions to a specific value of the charge, $q = (4\pi)^{-\frac{1}{2}}\,\eta_{00}$, there are still an infinite number of solutions for each of the discrete energies.

The final case, for which we can calculate the energy, is $\delta=0$, $K=\tau g^m$. The solutions here are given by (3.39) and (3.40) as

$$\vec{A} = \frac{m\tau}{c_n}\, r\, (g_n)^{m-1}\, \vec{I}_r$$

$$\phi = q\,\phi_{n0} \qquad (4.10)$$

$$g = g_n = \frac{c_n}{R^2} \quad .$$

Again, starting from the expression

$$E = \frac{1}{12\pi}\int\left((\vec{\nabla}g)^2 + 2(\vec{A}^2-\phi^2)g^2 \right) d^3x \quad , \qquad (A1.15)$$

we find, substituting in the solutions (4.10)

$$E = \frac{1}{12\pi}\int\left((\vec{\nabla}g_n)^2 - 2q^2\phi_{n0}^2 g_n^2 + \frac{2m^2\tau^2}{c_n^2}\, r^2\, g_n^{2m} \right) d^3x \qquad (4.11)$$

$$= \frac{\pi(4n^2-1)}{48|a|}\left(1-\frac{q^2}{2}\right) + \frac{2}{3}m^2\tau^2(c_n^2)^{m-1}\int_0^\infty \frac{r^4}{R^{4m}}\, dr \quad ,$$

where the previous result (4.7) has been used in evaluating the first two integrals. The remaining integral is

$$(4.12) \qquad I_3 \equiv \int \frac{r^4}{R^{4m}}\, dr = \frac{3\pi}{2} \frac{1}{|2a|^{4m-5}} \frac{(4m-7)!}{(2m-4)!(2m-1)!}$$

Substituting the expression (4.12) into (4.11) gives

$$(4.13) \qquad E = \frac{\pi(4n^2-1)}{48|a|}\left(1-\frac{q^2}{n^2}\right) + \frac{\pi\tau^2 c_n^{2m-2}}{|2a|^{4m-5}} \frac{m^2(4m-7)!}{(2m-4)!(2m-1)!}$$

Using the definition (4.7) for $E_0(n)$ we find

$$(4.14) \qquad \begin{aligned} E &= E_0(n)\ (1+2\tau^2) \quad , \text{ when } m=2, \text{ and} \\ &= E_0(n)\left[\, 1 + \frac{3\tau^2 n^2 m^2}{(n^2-1)} \frac{(4m-7)!}{(2m-4)!(2m-1)!} \left(\frac{4n^2-1}{32a^2}\right)^{m-2} \right] \text{ for } m \geq 3. \end{aligned}$$

The difference in the form for $m=2$, and $m\geq 3$, follows from the difference in the charge in the two cases. From (3.45), the charge, q, is given by $q^2=1+2\tau^2$ when $m=2$, whereas $q^2=1$ for $m\geq 3$.

Again in this example we find that the rest energy is positive. The last case considered has the property that, if the parameter τ, appearing in K(g), is unrestricted, there is a continuum of allowed rest energies[†]. This undesirable feature can be eliminated by giving τ some fixed numerical value. For any given τ, there is a discrete spectrum of energies corresponding to different values for n.

Note that if τ is different from zero, then the nodeless 'ground state', $n=1$, no longer has zero energy, if $m\geq 3$. In particular, from (4.13):

$$E(n=1, m\geq 3) = \frac{3\pi}{16} \frac{\tau^2 m^2 (4m-7)!}{(2m-4)!(2m-1)!} \left(\frac{3}{32a^2}\right)^{m-2} .$$

[†] Also, when $m=2$, there is a continuum of charge values.

We see that the non-zero vector potential, $\vec{A}$, resulting from the term $K\,\vec{\nabla}\cdot\vec{A}$, in the Lagrangian, can contribute to the energy, even though it does not produce an electric or magnetic field. This is a particular example of how the potential in a non-gauge-invariant theory can take on physical significance.

In the examples considered in this chapter, we have seen that, even in a classical theory, it is possible to have the energy restricted to a spectrum of discrete values. Also, we have found that the rest energies of all the particle-like solutions obtained, are positive. In the next chapter, we consider another fundamental property of elementary particles, namely, the angular momentum.

CHAPTER 5

ANGULAR MOMENTUM IN CLASSICAL FIELD THEORY

5.1 Total Angular Momentum

In the treatment of classical field theory, many authors† discuss the conservation of angular momentum. However, there seems to be only a minimal discussion of the circumstances under which it is useful to divide the total angular momentum into orbital and intrinsic, (spin), contributions. The main topic of this chapter is this division of the total angular momentum.

It follows from Noether's theorem* that the total angular momentum of a system is conserved if the Lagrangian is invariant under infinitesimal, homogeneous Lorentz transformations and does not depend explicitly on the coordinates, x_μ. The conserved total angular momentum has spatial components, J_k, given by

$$J_k = \tfrac{1}{2}\epsilon_{klm} \int_\sigma J_{lm\lambda}\, d\sigma_\lambda \quad , \tag{5.1}$$

where σ is a space-like hypersurface with a normal specified by $d\sigma_\lambda$. The angular momentum density, $J_{\mu\nu\lambda}$, to be defined shortly, satisfies the continuity equation

$$J_{\mu\nu\lambda,\lambda} = 0 \quad . \tag{5.2}$$

The relation (5.2) ensures that the definition (5.1) is independent of

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† These include Aharoni (1965), Barut (1964), Rzewuski (1967) and Soper (1976).

* See, for example, Soper (1976), p 105.

the choice of the surface σ. Taking σ to be a surface of constant time, one has

$$J_k = -\frac{i}{2}\,\varepsilon_{klm}\int J_{lm4}\,d^3x \quad . \tag{5.3}$$

The definition of the angular momentum density, $J_{\mu\nu\lambda}$, depends on the Lorentz transformation properties of the fields involved in the system of study. Define an infinitesimal homogeneous Lorentz transformation by

$$\begin{aligned} x_\mu \to x'_\mu &= x_\mu + \delta x_\mu \quad , \text{ with} \\ \delta x_\mu &= \varepsilon_{\mu\nu} x_\nu \quad ; \\ \varepsilon_{\mu\nu} &= -\varepsilon_{\nu\mu} \end{aligned} \tag{5.4}$$

The four-vector potential A_μ transforms according to

$$A_\mu(x_\lambda) \to A'_\mu(x'_\lambda) = A_\mu(x_\lambda) + \frac{i}{2}\,\varepsilon_{\sigma\rho}\,(S_{\sigma\rho})_{\mu\nu}\,A_\nu(x_\lambda) \quad , \text{ where} \tag{5.5}$$

$$(S_{\sigma\rho})_{\mu\nu} = -i\,(\,\delta_{\sigma\mu}\delta_{\rho\nu} - \delta_{\sigma\nu}\delta_{\rho\mu}\,) \quad . \tag{5.6}$$

For the systems we have studied, in which the Lagrangian, $\mathcal{L}$, is constructed from a scalar field, g, and the vector field A_μ, the angular momentum density is then defined as

$$J_{\mu\nu\lambda} = x_\mu T_{\nu\lambda} - x_\nu T_{\mu\lambda} + S_{\mu\nu\lambda} \quad , \text{ where} \tag{5.7}$$

$$S_{\mu\nu\lambda} \equiv i\,(S_{\mu\nu})_{\alpha\beta}\,A_\alpha\,\frac{\partial\mathcal{L}}{\partial A_{\beta,\lambda}} \tag{5.8}$$

$$= A_\mu\,\frac{\partial\mathcal{L}}{\partial A_{\nu,\lambda}} - A_\nu\,\frac{\partial\mathcal{L}}{\partial A_{\mu,\lambda}} \quad , \text{ using (5.6)} \quad . \tag{5.9}$$

Here $T_{\mu\nu}$ is the canonical energy-momentum tensor, given by

$$T_{\mu\nu} \equiv \delta_{\mu\nu}\mathcal{L} - A_{\sigma,\mu}\,\frac{\partial\mathcal{L}}{\partial A_{\sigma,\nu}} - g_{,\mu}\,\frac{\partial\mathcal{L}}{\partial g_{,\nu}} \quad . \tag{5.10}$$

For problems involving other tensor or spinor fields than A_μ, the transformation law analogous to (5.5) is used to define appropriate generalizations of the matrices $S_{\sigma\rho}$, which introduces additional terms into the definition (5.7) of the angular momentum density.

While the definition (5.7) is the one which arises directly from the invariance of the Lagrangian under homogeneous Lorentz transformations, there is another common definition of angular momentum, which we denote by $J^{\theta}_{\mu\nu\lambda}$. This second definition is based on the symmetric energy-momentum tensor $\theta_{\mu\nu}$, introduced by Belinfante (1940) and Rosenfeld (1940), which exists for any Lorentz invariant Lagrangian. $\theta_{\mu\nu}$ is related to the canonical energy-momentum tensor, $T_{\mu\nu}$, by

$$\theta_{\mu\nu} = \theta_{\nu\mu} \equiv T_{\mu\nu} + g_{\mu\nu\lambda,\lambda} \quad , \text{ where} \tag{5.11}$$

$$g_{\mu\nu\lambda} \equiv -\tfrac{1}{2}\left(S_{\mu\nu\lambda} + S_{\lambda\nu\mu} + S_{\lambda\mu\nu} \right), \tag{5.12}$$

$S_{\mu\nu\lambda}$ being the tensor introduced in (5.8). The symmetry property

$$S_{\mu\nu\lambda} = - S_{\nu\mu\lambda} \tag{5.13}$$

leads to the result

$$g_{\mu\nu\lambda} = - g_{\mu\lambda\nu} \tag{5.14}$$

which, in turn, ensures that

$$\theta_{\mu\nu,\nu} = T_{\mu\nu,\nu} + g_{\mu\nu\lambda,\lambda\nu} = 0 \quad , \quad (\text{using } T_{\mu\nu,\nu} = 0) \, . \tag{5.15}$$

Therefore, the energy-momentum

$$P^{\theta}_{\mu} \equiv \int_{\sigma} \theta_{\mu\nu} \, d\sigma_\nu \tag{5.16}$$

is conserved. Furthermore, since $\theta_{\mu\nu}$ and $T_{\mu\nu}$ differ by a divergence,

(5.17) $$P^{\theta}_{\mu} = P_{\mu} \equiv \int_{\sigma} T_{\mu\nu} d\sigma_{\nu} \quad ,$$

for fields which fall to zero sufficiently fast at spatial infinity.

Now define the angular momentum density, $J^{\theta}_{\mu\nu\lambda}$, by

(5.18) $$J^{\theta}_{\mu\nu\lambda} \equiv x_{\mu}\theta_{\nu\lambda} - x_{\nu}\theta_{\mu\lambda}$$

and the corresponding total angular momentum, J^{θ}_{k}, by

(5.19) $$J^{\theta}_{k} = \tfrac{1}{2}\epsilon_{klm} \int_{\sigma} J_{lm\lambda}\, d\sigma_{\lambda} = \epsilon_{klm} \int_{\sigma} x_{l}\theta_{m\lambda}\, d\sigma_{\lambda} \quad .$$

From the result (5.15) and the symmetry of $\theta_{\mu\nu}$, one can easily show that

(5.20) $$J^{\theta}_{\mu\nu\lambda,\lambda} = 0 \quad ,$$

which ensures that the definition (5.19) for J^{θ}_{k} is independent of the surface σ. Choosing σ to be a surface of constant time, we have

(5.21) $$J_{k} = -i\, \epsilon_{klm} \int x_{l}\theta_{m4}\, d^{3}x \quad .$$

We now compare the total angular momentum, J^{θ}_{k}, defined by (5.21), to the first definition, J_{k}, given by (5.3) and (5.7). Using the definition (5.11) of the symmetrized tensor $\theta_{\mu\nu}$, one finds from (5.21):

(5.22) $$J^{\theta}_{k} = -i\epsilon_{klm} \int x_{l}(T_{m4} + g_{m4\lambda,\lambda})\, d^{3}x \quad .$$

and from (5.3) and (5.7):

(5.23) $$J_{k} = -i\epsilon_{klm} \int x_{l}(T_{m4} + \tfrac{1}{2}S_{lm4})\, d^{3}x \quad .$$

Combine (5.22) and (5.23) to give

(5.24) $$J^{\theta}_{k} - J_{k} = -i\epsilon_{klm} \int (x_{l}g_{m4\lambda,\lambda} - \tfrac{1}{2}S_{lm4})\, d^{3}x \quad .$$

Using $g_{m44}=0$, (which follows from (5.14)), (5.24) can be written

$$J^{\theta}_k - J_k = -i\,\epsilon_{klm}\int\left[\,(x_l g_{m4n})_{,n} - g_{m4l} - \tfrac{1}{2}S_{lm4}\,\right]d^3x\ .$$

The last two terms in this expression cancel when one substitutes the definition of g_{m4l} from (5.12). This leaves

$$J^{\theta}_k - J_k = -i\,\epsilon_{klm}\int (x_l g_{m4n})_{,n}\,d^3x\ . \tag{5.25}$$

This last integral can be converted to a surface integral at spatial infinity. Therefore, if the fields fall off fast enough as $r\to\infty$, then the right-hand side of (5.25) will vanish and we find $J^{\theta}_k = J_k$. However, there may be cases in which the fields do not go to zero fast enough so one should always be aware that there is the possibility of a difference in the two definitions of the total angular momentum. In the discussion of the division of the total angular momentum into spin and orbital parts, we find it convenient to refer to the total angular momentum J_k, rather than J^{θ}_k. When the two definitions may not be equivalent, however, both forms are used.

5.2 Orbital and Spin Angular Momentum

The total angular momentum density, $J_{\mu\nu\lambda}$, is given, from (5.7), as

$$J_{\mu\nu\lambda} = x_\mu T_{\nu\lambda} - x_\nu T_{\mu\lambda} + S_{\mu\nu\lambda}\ . \tag{5.7}$$

Looking at the terms that appear in (5.7), we define

$$L_{\mu\nu\lambda} \equiv x_\mu T_{\nu\lambda} - x_\nu T_{\mu\lambda} \quad \text{and} \tag{5.26}$$

$$L_k = \tfrac{1}{2}\,\epsilon_{klm}\int_\sigma L_{lm\lambda}\,d\sigma_\lambda = \epsilon_{klm}\int_\sigma x_l T_{m\lambda}\,d\sigma_\lambda\ . \tag{5.27}$$

Also define

$$S_k \equiv \tfrac{1}{2}\,\epsilon_{klm} \int_\sigma S_{lm\lambda}\, d\sigma_\lambda = \epsilon_{klm} \int_\sigma A_l \frac{\partial \mathcal{L}}{\partial A_{m,\lambda}}\, d\sigma_\lambda \,, \tag{5.28}$$

where (5.9) has been used to obtain the second equality in (5.28).

Recall that the momentum of the system with canonical energy-momentum tensor $T_{\mu\nu}$, is given by

$$P_k = \int_\sigma T_{k\mu}\, d\sigma_\mu \,.$$

Therefore, the quantity $L_{jk\mu}$, defined by (5.26), has a form similar to the density of the angular momentum, $\vec{r}\times\vec{p}$, of classical particle dynamics. This similarity has led some authors to call L_k, as defined by (5.27), the orbital angular momentum of the system. The term, S_k, given by (5.28), is then identified as the intrinsic angular momentum, or spin, of the system. The total angular momentum, J_k, is

$$J_k = L_k + S_k \,.$$

Soper (1976), p 114, mentions that the volume integral of $S_{\mu\nu\lambda}$ is the operator for the spin in quantum field theory, lending support to the hypothesis that S_k is the spin.

However, there are several problems associated with these identifications for the orbital and spin angular momenta. In general, the definitions (5.27) and (5.28), of L_k and S_k, are not independent of the spacelike surface σ. So, for example, if σ is a surface of constant time, t, for one observer, a Lorentz-transformed observer cannot use his surface of constant time, t', and expect to calculate values for L_k and S_k which are consistent with those calculated by the first observer. One has the problem of deciding which hypersurface, σ, to use in the evaluation of

the integrals for L_k and S_k.

The condition for L_k and S_k to be independent of σ is †

(5.29) $$L_{lm\lambda,\lambda} = 0 \quad , \text{ or}$$

(5.30) $$S_{lm\lambda,\lambda} = 0 \quad .$$

(5.29) and (5.30) are also just the conditions necessary for the orbital and spin angular momenta, L_k and S_k, to be separately conserved. It is easy to show that (5.29) and (5.30) are satisfied if the canonical energy-momentum tensor is symmetric, for in that case we have, using the definition (5.21) of $L_{\mu\nu\lambda}$ and the result $T_{\mu\nu,\nu} = 0$:

(5.31) $$L_{\mu\nu\lambda,\lambda} = (x_\mu T_{\nu\lambda} - x_\nu T_{\mu\lambda})_{,\lambda} = T_{\nu\mu} - T_{\mu\nu} = 0 \quad .$$

Therefore, when the canonical tensor is symmetric the orbital and spin angular momenta (5.27) and (5.28) are well defined and conserved. This result has led Hilgevoord and Wouthuysen (1963) and Hilgevoord and de Kerf (1965), to propose new definitions for the spin and orbital angular momenta, so as to be conserved for free field systems, even in the cases where the canonical tensor of the usual formulation is not symmetric.* However, as of the 1965 paper, the method had been applied only to those systems for which the field equations could be expressed as Klein-Gordon equations subject to some subsidiary conditions. (The Dirac equation is of this type.)

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† (5.29) and (5.30) are not independent because $J_{\mu\nu\lambda} = L_{\mu\nu\lambda} + S_{\mu\nu\lambda}$ and $J_{\mu\nu\lambda,\lambda} = 0$.

* More correctly, the form of the definitions (5.27) and (5.28) is retained, but a Lagrangian giving rise to a non-symmetric energy-momentum tensor is replaced by a new Lagrangian, plus subsidiary conditions on the field equations, for which the canonical energy-momentum tensor is symmetric.

We now examine some other aspects of the spin and orbital angular momentum. Suppose that one has chosen the surface, σ, to be a surface of constant time for some observer, so that L_k and S_k are unambiguously defined. Then from (5.27) and (5.28) we have

$$L_k = -i\,\varepsilon_{klm} \int x_l\, T_{m4}\, d^3x \quad , \text{ and} \tag{5.32}$$

$$S_k = -i\,\varepsilon_{klm} \int A_l\, \frac{\partial \mathcal{L}}{\partial A_{m,4}}\, d^3x \quad . \tag{5.33}$$

Now look at the behaviour of L_k under translations. The orbital angular momentum, calculated in a reference frame translated by an amount a_μ, from a frame in which L_k is given by (5.32), is

$$L_k' = -i\,\varepsilon_{klm} \int x_l'\, T'_{m4}(x'_\mu)\, d^3x' \quad , \text{ where} \tag{5.34}$$

$$x'_\mu = x_\mu - a_\mu \quad . \tag{5.35}$$

All known fields are scalars under translations, so

$$T'_{\alpha\beta}(x'_\mu) = T_{\alpha\beta}(x_\mu) \quad . \tag{5.36}$$

Changing the integration variable in (5.34) to the unprimed coordinates, using $d^3x' = d^3x$, and substituting from (5.35) and (5.36), one has

$$\begin{aligned} L_k' &= -i\,\varepsilon_{klm} \int x_l'\, T'_{m4}(x'_\mu)\, d^3x \\ &= -i\,\varepsilon_{klm} \int (x_l + a_l)\, T_{m4}(x_\mu)\, d^3x \quad . \\ &= L_k - i\,\varepsilon_{klm}\, a_l \int T_{m4}\, d^3x \quad . \end{aligned} \tag{5.37}$$

In the rest frame of a particle, from Appendix 1, we know that the integral appearing in (5.37) vanishes, since it is just the three-momentum, P_m, to within a factor. Such a rest frame exists when the system is described

by time-independent localized fields. From (5.37) we see that for such solutions $L_k' = L_k$. In such a case, L_k is translationally invariant, or equivalently, we may say that the orbital angular momentum is independent of the point about which it is calculated. This result is consistent with our expectation that the orbital angular momentum should vanish in all reference frames in which the three-momentum is zero. However, we have not been able to show that the integral (5.32) for L_k does vanish for all time-independent localized solutions. If L_k is non-zero in any zero momentum frame, it would describe what would be called an intrinsic, rather than an orbital property. If one could show that $L_k=0$, in every such zero momentum frame, then the interpretation of L_k as orbital angular momentum would be much more satisfactory than it presently appears to us.

Another point illustrating that the division, of the total angular momentum into spin and orbital parts, is not clear cut, is that for a variety of Lagrangians one can show that the integrals for L_k and S_k are proportional for all localized time-independent solutions. As an example, consider the class of Lagrangian constructed from the following scalar quantities:

$$\mu = \tfrac{1}{2} A_{\sigma,\rho} A_{\rho,\sigma} \qquad \chi = \tfrac{1}{2} g_{,\sigma} g_{,\sigma}$$

$$\nu = \tfrac{1}{2} A_{\sigma,\rho} A_{\sigma,\rho} \qquad \xi = g_{,\sigma} A_{\sigma} \tag{5.38}$$

$$\omega = A_{\sigma,\sigma} \qquad \zeta = A_{\sigma} A_{\sigma}$$

Let the Lagrangian density have the form

$$\mathcal{L} = U\mu + V\nu + \overline{\mathcal{L}}(\omega,\chi,\xi,\zeta,g) \ , \tag{5.39}$$

where U and V are constants and $\mathcal{L}$ is an arbitrary differentiable function of its arguments. The general form (5.39) includes all the Lagrangians studied in Chapter 3 for which exact solutions were found. The ordinary electrodynamics Lagrangian, $\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu}{}^2$, is the case $U=-V=\frac{1}{4\pi}$ and $\mathcal{I}=0$. In Appendix 4 we derive the following results for the angular momenta of time-independent, sufficiently localized solutions of the field equations derived from Lagrangians of the form (5.39):

$$\vec{S} = U \int \vec{E}\times\vec{A}\, d^3x$$

(A4.17)
$$\vec{L} = (2V-U) \int \vec{E}\times\vec{A}\, d^3x$$

$$\vec{J} = \vec{L} + \vec{S} = 2V \int \vec{E}\times\vec{A}\, d^3x \quad ,$$

based on the definitions (5.32) and (5.33), and $\vec{E}=-\vec{\nabla}\phi$

As indicated previously, the interpretation of $\vec{L}$ as the orbital angular momentum is satisfactory only if it vanishes for static, localized solutions. We see from (A4.17) that, with the exception of the case U=2V, the vanishing of $\vec{L}$ implies that the spin angular momentum is also zero. So, our conclusion is that either there is no time-independent system of the form (5.39) which can describe a localized particle with non-zero spin, or the identification of $\vec{L}$, given by (5.31), as the orbital angular momentum, is not justified.

The results (A4.17) also illustrate a problem mentioned by Belinfante (1940) in connection with the division of the total angular momentum into spin and orbital parts: two different Lagrangians, generating the same field equations, can give a different splitting of the total angular momentum. To take a specific example, consider the following two Lagrangians:

$$\mathcal{L}_I = \frac{1}{4\pi}\left(\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(A_{\sigma,\sigma})^2 + \mathcal{L}'\right) \tag{5.40}$$

$$\mathcal{L}_{II} = -\frac{1}{4\pi}\left(\frac{1}{2}(A_{\sigma,\rho})^2 + \mathcal{L}'\right) , \tag{5.41}$$

where $\mathcal{L}' = \mathcal{L}'(g_{,\sigma}g_{,\sigma}\,;\, g_{,\sigma}A_\sigma\,;\, A_\sigma A_\sigma\,;\, g)$.

The field equations derived from both Lagrangians are

$$A_{\sigma,\rho\rho} = \frac{\partial\mathcal{L}'}{\partial A_\sigma}$$

$$\partial_\rho\left(\frac{\partial\mathcal{L}'}{\partial g_{,\rho}}\right) = \frac{\partial\mathcal{L}'}{\partial g} .$$

The Lagrangian, $\mathcal{L}_I$, given by (5.40), corresponds to $U=-V=\frac{1}{4\pi}$ in (5.39), whereas $\mathcal{L}_{II}$ corresponds to $U=0$; $V=-\frac{1}{4\pi}$. We find from (A4.17) that the angular momenta for the Lagrangians $\mathcal{L}_I$ and $\mathcal{L}_{II}$ are

$$\begin{aligned} \vec{S}^I &= \vec{S} & \vec{S}^{II} &= 0 \\ \vec{L}^I &= -3\vec{S} & \vec{L}^{II} &= -2\vec{S} \\ \vec{J}^I &= -2\vec{S} & \vec{J}^{II} &= -2\vec{S} \end{aligned} \tag{5.42}$$

where $S = \frac{1}{4\pi}\int \vec{E}\times\vec{A}\, d^3x$; $\vec{E} = -\vec{\nabla}\phi$. Although the total angular momenta J^I and J^{II} are equal, (5.42) shows that unless all the angular momenta are zero, the division into spin and orbital pieces depends on the form of the Lagrangian.

We regard the problem of defining the spin and orbital angular momenta in classical field theory as remaining unsolved. We have not been able to show that a common definition of the orbital angular momentum vanishes in the rest frame of a particle, as it should. The definition suffers from a lack of criteria to determine what hypersurface to use in evaluating the relevant integrals. There is also an ambiguity arising from a dependence on the form of the Lagrangian. A proportion-

ality between the orbital and spin angular momenta for time-independent localized solutions in a variety of cases, and the translational invariance of the orbital angular momentum, in the same cases, suggests a possible 'intrinsic' component to what as been defined as the orbital angular momentum. Until these difficulties are resolved, to us, it seems that the best approach is to consider the total angular momentum, $\vec{J}$, in a zero-momentum frame, (rest frame), as the intrinsic angular momentum of the particle. One then still has to resolve the problem of the meaning of the separate conservation of L_k and S_k when the canonical energy-momentum tensor is symmetric.

5.3 Angular Momentum for Exact Solutions

We now calculate the angular momentum of the exact solutions found in Chapter 3. The Lagrangian for the systems of Chapter 3 is

$$(2.65)\quad \mathcal{L} = -\frac{1}{4\pi}\left[\frac{1}{2}(g_{,\mu})^2 + \frac{1}{4}F_{\mu\nu}{}^2 + \frac{\delta}{2}(A_{\mu,\mu})^2 + K(g)A_{\mu,\mu} - g^2A_\mu^2 + M(g)\right].$$

This is of the form (5.39), with $U=-V=\frac{1}{4\pi}$. In terms of the variables μ,ν,ω,χ,ξ, and ζ introduced in (5.38), $\mathcal{L}$ is

$$(5.43)\qquad \mathcal{L}_I = -\frac{1}{4\pi}\left[\nu - \mu + \frac{\delta}{2}\omega^2 + K(g)\omega + \chi - g^2\zeta + M(g)\right].$$

Another Lagrangian leading to the same field equations is

$$(5.44)\qquad \mathcal{L}_{II} = -\frac{1}{4\pi}\left[\nu + (\delta-1)\mu + K\omega + \chi - g^2\zeta + M(g)\right],$$

which corresponds to $U=\frac{1-\delta}{4\pi}$ and $V=-\frac{1}{4\pi}$. When $\delta=0$, $\mathcal{L}_I = \mathcal{L}_{II}$, but when $\delta=1$, we will have to distinguish between $\mathcal{L}_I$ and $\mathcal{L}_{II}$ when calculating angular momenta.

First consider the case $\delta=0$. From (5.43) in (A4.5) and (A4.7),

we find the 'orbital' angular momentum

$$L_k = -\frac{1}{4\pi}\,\varepsilon_{klm}\int x_l\left(A_{n,m}\phi_{,n} - K\phi_{,m}\right)\,d^3x \tag{5.44}$$

and the 'spin' angular momentum

$$\vec{S} = \frac{1}{4\pi}\int \vec{E}\times\vec{A}\,d^3x \quad ; \quad \vec{E} = -\vec{\nabla}\phi \quad . \tag{5.45}$$

The total angular momentum, $\vec{J}$, is given by

$$\vec{J} = \vec{L} + \vec{S} \quad . \tag{5.46}$$

When $\delta=0$ and $K=0$, the only solutions we were able to find are given by (3.26). The vector potential, $\vec{A}=0$, for these solutions. From (5.44), (5.45), and (5.46) we see that if $K=0$ and $\vec{A}=0$, then

$$\vec{L} = \vec{S} = \vec{J} = 0 \quad .$$

When $\delta = 0$ and $K\neq 0$, and specifically for $K=\tau g^m$, we found solutions of the form $\vec{A}=A(r)\,\vec{I}_r$, with A given by (3.39), and $\phi=\phi(r)$, given by (3.40). Since $\vec{E}=-\vec{\nabla}\phi=-\frac{d\phi}{dr}\vec{I}_r$ for these solutions, $\vec{E}$ is parallel to $\vec{A}$. We find from (5.45), using this result, that $\vec{S}=0$. Also, since $\vec{A}=A(r)\,\vec{I}_r$ and $\phi=\phi(r)$, one has

$$A_{n,m} = \left(\frac{x_n A}{r}\right)_{,m} = \frac{A}{r}\delta_{mn} + \frac{x_n x_m}{r}\frac{d}{dr}\left(\frac{A}{r}\right) \quad \text{and}$$

$$\phi_{,n} = \frac{x_n}{r}\frac{d\phi}{dr} \quad . \tag{5.47}$$

Substituting (5.47) into (5.44) and using $\varepsilon_{klm}=-\varepsilon_{kml}$, one finds that $\vec{L}=0$. Combining this with our previous result for $\vec{S}$ gives $\vec{J}=\vec{L}=\vec{S}=0$. Therefore, for all the exact solutions found with $\delta=0$, the angular momenta are all zero. However, in the particular case where $K=\tau g^2$, (3.39) gives

$A(r)=\frac{2\tau r}{R^2}$, so A goes to zero as r^{-1}, as $r\to\infty$. In this case there is the possibility that the definitions of the angular momentum based on the canonical and symmetrized tensors, may be different. From (5.25), these definitions are related by

$$J_k^\theta - J_k = -i\,\varepsilon_{klm}\int (x_l g_{m4n})_{,n}\, d^3x \quad . \tag{5.25}$$

One finds from (5.9) that

$$S_{\mu\nu\lambda} = \frac{1}{4\pi}\left(A_\mu(F_{\lambda\nu}-K\delta_{\nu\lambda}) - A_\nu(F_{\mu\lambda}-K\delta_{\mu\lambda})\right) \quad . \tag{5.48}$$

Using (5.48) in the definition (5.12) of $g_{\mu\nu\lambda}$, we get

$$g_{\mu\nu\lambda} = \frac{1}{4\pi}\left(A_\mu F_{\lambda\nu} + K(A_\lambda\delta_{\mu\nu}-A_\nu\delta_{\mu\lambda})\right) \quad . \tag{5.49}$$

Substituting (5.49) into (5.25) and using the results (5.47), one finds that the right-hand side of (5.25) vanishes, leaving $J_k^\theta = J_k$. Therefore, we may say unambiguously that the angular momentum is zero for all the solutions with $\delta=0$.

Next we consider those solutions with $\delta=1$ and $K=0$, given by (3.18) and (3.29). In this case, using the Lagrangian, $\mathcal{L}_I$, given by (5.43), we find from (A4.5), that the static angular momenta are

$$L_k^I = -\frac{1}{4\pi}\,\varepsilon_{klm}\int x_l(A_{j,m}\phi_{,j}-A_{j,j}\phi_{,m})\, d^3x \tag{5.50}$$

and from (A4.7)

$$\vec{S}^I = \frac{1}{4\pi}\int \vec{E}\times\vec{A}\, d^3x \quad . \tag{5.51}$$

However, using the Lagrangian, $\mathcal{L}_{II}$, given by (5.44), we find from (A4.5) and (A4.7), that

$$\vec{L}^{II} = \vec{S}^{II} = 0 \ , \text{ so } \vec{J}^{II} = 0 \quad .$$

With considerably more effort, we can show that $\vec{L}^I$ and $\vec{S}^I$, as given by (5.50) and (5.51), also vanish. To see this start by manipulating the expression (5.50) for L_k^I into the form

$$(5.52)\quad L_k^I = -\frac{1}{4\pi}\,\varepsilon_{klm}\int\left[\,(x_l A_j \phi_{,j})_{,m} - (x_l A_j \phi_{,m})_{,j} + A_l \phi_{,m}\,\right] d^3x \quad ,$$

or in vector form, using (5.51) in the last term

$$(5.53)\quad \vec{L}^I = -\frac{1}{4\pi}\int\left[\,\vec{\nabla}\times\left((\vec{A}\cdot\vec{E})\vec{r}\right) + (A_j\,\vec{r}\times\vec{E})_{,j}\,\right] d^3x \;-\; \vec{S}^I \quad ,$$

$$= \lim_{r\to\infty}\left(\frac{1}{4\pi}\int_S \vec{A}\cdot\vec{E}\;\vec{r}\times\vec{n}\;dS \;-\frac{1}{4\pi}\int_S \vec{r}\times\vec{E}\;\vec{A}\cdot\vec{n}\;dS\right) - \vec{S}^I \quad ,$$

where $\vec{n}$ is the outward normal to the surface S. We can take S to be a sphere of radius r, so that $\vec{n} = \vec{1}_r$. As a result, $\vec{r}\times\vec{n} = 0$ and the first integral in (5.53) vanishes. In the limit $r\to\infty$, the electric field, $\vec{E}$, for the solution (3.18) is radial so that $\vec{r}\times\vec{E}\to 0$. Therefore, the second integral in (5.53) vanishes as well, leaving

$$(5.54)\quad \vec{L}^I = -\,\vec{S}^I \quad .$$

This implies that the total angular momentum $\vec{J}^I = \vec{L}^I + \vec{S}^I = 0$. We can go one step further, to show that $\vec{L}^I$ and $\vec{S}^I$ vanish separately. From (A4.6) in Appendix 4:

$$(A4.6)\quad S_k^I = \frac{1}{4\pi}\,\varepsilon_{klm}\int A_l \phi_{,m}\, d^3x \quad , \text{ so}$$

$$(5.55)\quad S_k^I = \frac{1}{8\pi}\,\varepsilon_{klm}\int\left[\,\left(x_l(\phi A_{m,j} - \phi_{,j}A_m)\right)_{,j} - (\phi A_m)_{,l}\,\right] d^3x$$

$$+\frac{1}{8\pi}\,\varepsilon_{klm}\int x_l(\phi_{,jj}A_m - \phi A_{m,jj})\, d^3x \quad ,$$

where the second expression follows as an identity from the first, as can be seen by expanding the derivatives and simplifying. The integrand of

the second integral in (5.55) can be shown to vanish directly from the field equations (3.32.1) and (3.32.2). The first integral in (5.55) can be transformed to a surface integral at spatial infinity. Using the explicit form of the solutions (3.18) and (3.21), in the limit $r\to\infty$, one can show that this surface integral vanishes also, so that $S_k^I = 0$. Therefore, from (5.54) we have

$$\vec{S}^I = \vec{L}^I = \vec{J}^I = 0 \quad . \tag{5.56}$$

Before stating that the angular momentum of the solutions is zero we now check that the definition of the total angular momentum, J_k^θ, based on the symmetrized energy-momentum tensor, gives the same zero result. From (5.25)

$$J_k^\theta - J_k = -i\,\varepsilon_{klm} \int (x_l g_{m4n})_{,n}\, d^3x \quad , \tag{5.25}$$

with $g_{\mu\nu\lambda}$ defined by (5.12) and (5.8). Converting the integral (5.25) to a surface integral at spatial infinity and using the form of the solutions (3.18) and (3.29), in the limit $r\to\infty$, one finds that

$$J_k^\theta - J_k = 0 \quad . \tag{5.57}$$

The detailed calculation of the result (5.57) is rather lengthy and will not be presented here. The result (5.57) holds for both the Lagrangians $\mathcal{L}_I$ and $\mathcal{L}_{II}$.

In summary, we conclude that all the angular momenta, $\vec{S}$, $\vec{L}$, and $\vec{J}$, are zero for the exact solutions we have found. The difficulty of interpretation which would arise from a non-zero orbital angular momentum is therefore not present. Also, we have seen that the two definitions of the total angular momentum, based on the canonical and symmetrized energy-

momentum tensors, respectively, agree for these solutions. Furthermore, the two Lagrangians studied which led to the same field equations, also gave the result for the angular momentum.

CHAPTER 6

STABILITY ANALYSIS

6.1 Two Criteria For Stability

The question of whether or not solutions of the field equations are stable against small perturbations is one which frequently arises in the study of nonlinear classical field theories. It is not clear what stability requirements need to be imposed at the classical level. One possibility is that there need be no restrictions, based on stability, on the acceptability of a solution to a classical field theory. One would then rely on quantization of the classical solution to introduce some element of stability. In this regard one is reminded of Bohr's original quantum treatment of the hydrogen atom, in which a classical system with the same structure would collapse due to the emission of radiation.

While, experimentally, there are no known mesons which are absolutely stable, they do exist for a finite time before decaying. This suggests that an appropriate classical model would be one in which the solutions are metastable, that is, the time scale for the decay of a particle-like solution is long compared to some other characteristic time of interest. An example of a system for which metastable solutions exist, has been given by Rosen (1965).

A third alternative in the treatment of stability of solutions of classical field theories is to require that the particles be absolutely stable against small perturbations. In this chapter we study two criteria that have been used in the literature for deciding stability. The

first of these tests is based on the premise that the energy of a stable static solution should be a minimum with respect to static variations of the fields, about that solution. The second test requires for stability, of a static solution, that any slightly different solution oscillate about, rather than diverge from, the given static solution as time increases. After presenting a simple example to illustrate the two types of analysis and the relation between them, we attempt a similar study of one of the systems for which exact solutions were found in Chapter 3.

We first look at an example of the application of the minimum energy requirement with respect to static variations about a static solution. Using this criterion, Derrick (1964) showed the instability of solutions for a class of Lagrangian constructed from a real scalar field, ψ. Specifically, consider the following Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\left[(\psi_{,\mu})^2 + \Gamma(\psi) \right] \quad , \tag{6.1}$$

where Γ is an arbitrary function of the scalar field, ψ. The field equation is

$$\nabla^2\psi - \frac{\partial \psi^2}{\partial t^2} = \frac{d\Gamma}{d\psi} \quad . \tag{6.2}$$

The rest energy for a static real solution $\psi = \psi_0(r)$ of the equation (6.2) is

$$E(\psi_0) = -\int \mathcal{L}\, d^3x = \int \left(\tfrac{1}{2}(\vec{\nabla}\psi_0)^2 + \Gamma(\psi_0) \right) d^3x \quad . \tag{6.3}$$

Now calculate the value of the integral (6.3) when ψ_0 is replaced by a perturbed function, $\hat{\psi}$, given by

$$\hat{\psi}(\vec{r}) = \psi_0(\vec{r}) + \varepsilon\, \psi^1(\vec{r}) + O(\varepsilon^2) \quad , \tag{6.4}$$

where ε is a parameter sufficiently small that the second order terms can be neglected. The function, $\psi^1(r)$ is arbitrary except that it is sufficiently well behaved that it can be regarded as a perturbation to the term ψ_0. Expanding $E(\hat{\psi})$ in a Taylor series in ε we find from (6.3) that

$$E(\hat{\psi}) = E(\psi_0+\varepsilon\psi^1) = E(\psi_0) + \varepsilon\ \delta E + \varepsilon^2\ \delta^2 E + O(\varepsilon^3) \quad , \tag{6.5}$$

where

$$\delta E = \left.\frac{dE}{d\varepsilon}\right|_{\varepsilon=0} \quad \text{and} \tag{6.6}$$

$$\delta^2 E = \frac{1}{2}\left.\frac{d^2E}{d\varepsilon^2}\right|_{\varepsilon=0} \quad . \tag{6.7}$$

Using the fact that ψ_0 satisfies the field equation (6.2), one finds that $\delta E=0$. This is simply a consequence on the fact that the field equation was derived from a Lagrangian through a variational principle. Using (6.3) for the energy, one finds from the definition (6.7), that

$$\delta^2 E = \frac{1}{2}\int \left[(\vec{\nabla}\psi^1)^2 + (\psi^1)^2 \left.\frac{d^2\Gamma}{d\psi^2}\right|_{\psi=\psi_0} \right] d^3x \quad . \tag{6.8}$$

Now, following Derrick, if, as a particular case, we define the perturbed function $\hat{\psi}$, by a scale transformation of the original solution ψ_0:

$$\hat{\psi}(\vec{r}) \equiv \psi_0(\alpha\vec{r}) \quad ; \alpha>0 \quad , \tag{6.9}$$

($\alpha=1+\epsilon$ corresponds to $\psi^1(\vec{r}) = \vec{r}\cdot\vec{\nabla}\psi_0 = r\frac{\partial\psi_0}{\partial r}$)

then one finds from (6.8), or more easily by making a change of variable in (6.3), that

$$\delta^2 E = -\frac{1}{2}\int (\vec{\nabla}\psi_0)^2\, d^3x \quad . \tag{6.10}$$

Because the second variation, (6.10), of E is negative, (for $\psi_0\neq 0$), the

solution, ψ_0, does not provide a minimum of the energy functional $E(\psi)$, with respect to variations of the form (6.4). The solution is therefore said to be unstable.

Using the same example, (6.1), we now follow the treatment of Rosen (1965), in examining a second criterion for testing stability. Given a time-independent solution, $\psi_0(r)$, of the field equation (6.2), we assume that there exists a time-dependent real solution, $\tilde{\psi}(\vec{r},t)$, which differs slightly from ψ_0. Now we imagine that due to some transient external influence, the system described by the field ψ_0 is disturbed so as to be replaced by a system in which the field has the value $\tilde{\psi}$. The system is stable if all such solutions, $\tilde{\psi}$, remain near the original solution, ψ_0, as time increases. To study this possibility assume the following specific form for $\tilde{\psi}$:

$$\tilde{\psi}(\vec{r},t) = \psi_0(\vec{r}) + \varepsilon\, \psi_0^1(\vec{r}) \cos\omega t + O(\varepsilon^2) \quad , \tag{6.11}$$

where, again, ε is a parameter which is sufficiently small that the terms of second order can be neglected. The function, $\psi_0^1(\vec{r})$, giving the spatial dependence of the perturbation, is determined by the requirement that $\tilde{\psi}$ be a solution of the field equation (6.2). We will demand only that (6.2) be satisfied only to first order in ε.

We assume that ω is real or purely imaginary, so that $\psi_0^1(\vec{r})$ is a real function. If a solution of the form (6.11) exists for ω purely imaginary, then as the time, t, increases, $\tilde{\psi}$ will eventually differ from ψ_0 by a significant amount. In this case we say that the solution ψ_0 is unstable. On the other hand, if ω is real, the solution $\tilde{\psi}$ performs small oscillations about ψ_0. In this case the solution is stable against perturbations of the particular form (6.11), but we cannot conclude that the

system is actually stable because there may exist solutions into which ψ_0 could evolve, which are not of the form (6.11).

Substituting the proposed solution (6.11) into the field equation (6.2), one finds that it is satisfied to first order in ε if ψ_0^1 satisfies the equation:

$$\nabla^2\psi_0^1 - \left.\frac{\partial^2\Gamma}{\partial\psi^2}\right|_{\psi=\psi_0} \psi_0^1 = -\omega^2\,\psi_0^1 \quad . \tag{6.12}$$

It is important to note that the existence of a solution, ψ_0^1, to (6.12) does not guarantee that $\tilde{\psi}$, given by (6.11) is a solution of the nonlinear system (6.2) for any non-zero value of the expansion parameter ε. This is because we have considered only first order terms in ε. Determination of the conditions under which a solution, $\tilde{\psi}$, of the form (6.11) exists to the full nonlinear system (6.2) is a difficult mathematical problem that we have not considered. Assuming that such solutions do exist, the question of stability is thus reduced to determining the sign of the eigenvalue ω^2, in the eigenvalue problem, (6.12), for ψ_0^1.

Rosen indicates that the equation (6.12) can be derived from a variational principle:

$$\delta\omega^2 = 0 \ ^{\dagger}, \text{ with}$$

$$\omega^2 \equiv \omega^2\left(\psi_0^1\right) \equiv \frac{\int \left[\left(\vec{\nabla}\psi_0^1\right)^2 + \left.\frac{\partial^2\Gamma}{\partial\psi^2}\right|_{\psi=\psi_0} \left(\psi_0^1\right)^2 \right] d^3x}{\int \left(\psi_0^1\right)^2 d^3x} \quad . \tag{6.13}$$

---

† By $\delta\omega^2$ we mean $\frac{d}{d\varepsilon}\ \omega^2\left(\psi_0^1+\varepsilon\psi_0^2\right)\Big|_{\varepsilon=0}$ for an arbitrary variation ψ_0^2, with $\omega^2\left(\psi_0^1\right)$ given by (6.13).

Note that the form of ω^2 given by (6.13) is just that obtained by multiplying (6.12) by ψ_0^1 and integrating. The condition $\delta\omega^2 = 0$ means that ω^2 is an extremum when evaluated at ψ_0^1. It may further be shown from the properties of Sturm-Liouville equations,† that this extremum is a minimum. Therefore, assuming that a solution ψ_0^1 of (6.12) exists, then if (6.13) is evaluated for some trial function ψ^1 instead of ψ_0^1, and it is found that ω^2 is negative, then ω^2 will also be negative for the exact solution, ψ_0^1. Taking the trial function ψ^1 to be defined by a radial scale variation of the solution ψ_0, as in (6.9), one finds that ω^2 given by (6.13) is negative. This follows from our previous result (6.10) because the numerator in (6.13), evaluated at ψ^1, is just $\delta^2 E$, given by (6.8) and (6.10). Therefore, again, we conclude that the solution ψ_0 is unstable.

Using both the (static) criterion for instability, that $\delta^2 E<0$ for arbitrary static variations about ψ_0, and the (dynamic) criterion, that $\omega^2<0$ for time-dependent perturbations of ψ_0 satisfying the field equations, we found that static localized solutions, ψ_0, to the equation (6.2), are unstable. In this example there is a very close connection between the two tests of stability. Comparing (6.8) to (6.13) we see that if the static variation ψ^1 is equal to the function ψ_0^1 appearing in the time-dependent solution (6.11), then the following relation is obtained:

$$\omega^2(\psi_0^1) = \frac{2\,\delta^2 E(\psi_0^1)}{\int (\psi_0^1)^2\, d^3x} \,. \tag{6.14}$$

---

† Sturm-Liouville differential equations for several independent variables are discussed by Hellwig (1967).

Since ψ_0^1 is a real function, (6.14) shows that δ^2E and ω^2 have the same sign, when evaluated in terms of ψ_0^1. It is important to note that this identity of sign has been shown only for ψ_0^1 satisfying the condition (6.12) that $\tilde{\psi}$ given by (6.11) be a solution to the field equation (6.2) to first order in ϵ. However, the function

$$\psi^1 = r\frac{\partial \psi_0}{\partial r} \quad , \tag{6.15}$$

corresponding to the static scale variation (6.9), does not satisfy (6.12), in general. Thus, even though it happens that $\delta^2E(\psi^1)$ is negative for ψ^1 given by (6.15), we can only relate this result to a variational approximation, $\omega^2(\psi^1)$, to the exact value of $\omega^2 = \omega^2(\psi_0^1)$.

The nature of the stability of a particle is a statement regarding the evolution of a system with time. Therefore, we regard the 'dynamical' test of stability based on the sign of ω^2 as more easily understood than the 'static' test based on δ^2E. From this point of view, the relation (6.14) may be regarded as a justification for the use of δ^2E in determining stability. However, there are situations in which δ^2E cannot be directly related to ω^2, as indicated in the previous paragraph, and as will be seen when we apply these tests to another example. In these cases, unless there is some further argument in favor of deciding stability from δ^2E, we feel that the sign of ω^2 is the more meaningful test.

6.2 Stability Analysis For A System From Chapter 3

We now attempt to examine the stability of one of the simplest systems of chapter 3, namely, the case δ=K=0, in the Lagrangian (3.1). The full, time-dependent Lagrangian is:

$$\mathcal{L} = -\frac{1}{4\pi}\left[\frac{1}{4}F_{\mu\nu}{}^2 + \frac{1}{2}(g_{,\mu})^2 - g^2A_\mu^2 + M(g)\right] . \tag{6.16}$$

The corresponding field equations are, written out in detail:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) + \vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right) + \frac{\partial^2\vec{A}}{\partial t^2} - 2g^2\vec{A} = 0$$

$$\nabla^2\phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) + 2g^2\phi = 0 \tag{6.17}$$

$$\nabla^2 g - \frac{\partial^2 g}{\partial t^2} + 2(\vec{A}^2-\phi^2)g - N(g) = 0 ,$$

where $N(g) = \frac{dM}{dg}$.

Static solutions to (6.17) for $\phi(r)$ and $g(r)$, with $\vec{A}=0$, are given by (3.26). First look at the energy and its second static variation, δ^2E, for solutions of this type. From Appendix 1, the energy of solutions to the field equations (6.17) in the static case, is

$$E\left(g,\phi,\vec{A}\right) = -\int \mathcal{L}\, d^3x \tag{6.18}$$
$$= \frac{1}{4\pi}\int\left[\frac{1}{2}(\vec{\nabla}\times\vec{A})^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}(\vec{\nabla}g)^2 + (\phi^2-\vec{A}^2)g^2 + M\right] d^3x .$$

To first order in a small parameter ε, define

$$\hat{\phi}(\vec{r}) \equiv \phi(\vec{r}) + \varepsilon\,\phi'(\vec{r})$$

$$\hat{g}(\vec{r}) \equiv g(\vec{r}) + \varepsilon\, g'(\vec{r}) \tag{6.19}$$

$$\hat{\vec{A}}(\vec{r}) \equiv \vec{A}(\vec{r}) + \varepsilon\,\vec{A}'(\vec{r}) ,$$

where $\phi(\vec{r})$, $g(\vec{r})$, and $\vec{A}(\vec{r})$ are a static solution (for example, those given by (3.26)) to the field equations (6.17). Defining δ^2E by (6.7), that is,

$$\delta^2E = \frac{1}{2}\left.\frac{d^2E}{d\varepsilon^2}\right|_{\varepsilon=0} , \tag{6.7}$$

we find the following expression for δ^2E:

$$(6.20)\qquad \delta^2 E(\hat{g},\hat{\phi},\hat{\vec{A}}) = \frac{1}{8\pi}\int \Big[(\vec{\nabla}\times\vec{A}')^2 - (\vec{\nabla}\phi')^2 + (\vec{\nabla}g')^2 + \Big(\frac{dN}{dg} + 2(\phi^2 - \vec{A}^2)\Big)(g')^2 + 2g^2\big((\phi')^2 - (\vec{A}')^2\big) + 8gg'(\phi\phi' - \vec{A}\cdot\vec{A}')\Big]\, d^3x.$$

Because of the complexity of (6.20), we have not been able to determine the sign of $\delta^2 E$ for any general form of the perturbations g', ϕ', and $\vec{A}'$. However, in the particular case of the exact solutions (3.26), we were able to do an explicit calculation of $\delta^2 E$ for a static variation of the form[†]

$$(6.21)\qquad \begin{aligned} \phi(\vec{r}) \to \hat{\phi}(\vec{r}) &= s\phi(s\vec{r}) \quad ; \; s>0 \\ g(\vec{r}) \to \hat{g}(\vec{r}) &= sg(s\vec{r}) \quad . \end{aligned}$$

This variation is chosen rather than a simple scale variation because it preserves the value of the charge (i.e. the $\frac{1}{r}$ coefficient in an expansion of ϕ), whereas the scale variation does not. Writing $s=1+\varepsilon$, where $\varepsilon \ll 1$, comparison with (6.19) shows that this corresponds to

$$\begin{aligned} \phi'(\vec{r}) &= \phi(\vec{r}) + \vec{r}\cdot\vec{\nabla}\phi \\ g'(\vec{r}) &= g(\vec{r}) + \vec{r}\cdot\vec{\nabla}g \quad . \end{aligned}$$

One finds from (6.7) and (6.18) that

$$(6.22)\qquad E \equiv \frac{1}{2}\frac{d^2E}{d\varepsilon^2}\bigg|_{\varepsilon=0} = \frac{1}{2}\frac{d^2E}{ds^2}\bigg|_{s=1} = -\frac{3}{n+1}E_0(n) \quad ; \; n=1,2,3,\ldots \; ,$$

where $E_0(n)$ is the (non-negative) energy given by (4.7). In the case of the 'ground state', $n=1$, the energy $E_0(n=1)=0$, so that $\delta^2 E=0$, but for $n\geq 2$, $E_0(n)>0$ so $\delta^2 E<0$. Therefore, with the exception of the solution for $n=1$, if the sign of $\delta^2 E$ is a valid test for stability, then we conclude that all the exact solutions are unstable.

[†] This is sometimes called a dilation transformation.

Next consider time-dependent perturbations of the solutions $g(\vec{r})$, $\phi(\vec{r})$, and $\vec{A}(\vec{r})$. To first order in ε, let

$$\tilde{\phi}(\vec{r},t) \equiv \phi(\vec{r}) + \varepsilon \left(\phi^1(\vec{r})\, e^{i\omega t} + \phi^1(\vec{r})^*\, e^{-i\omega^* t} \right)$$

$$\tilde{g}(\vec{r},t) \equiv g(\vec{r}) + \varepsilon \left(g^1(\vec{r})\, e^{i\omega t} + g^1(\vec{r})^*\, e^{-i\omega^* t} \right) \tag{6.23}$$

$$\tilde{\vec{A}}(\vec{r},t) \equiv \vec{A}(\vec{r}) + \varepsilon \left(\vec{A}^1(\vec{r})\, e^{i\omega t} + \vec{A}^1(\vec{r})^*\, e^{-i\omega^* t} \right),$$

where g^1, ϕ^1 and $\vec{A}^1$ are arbitrary, non-singular complex valued functions which drop off at least as fast as g, ϕ, and $\vec{A}$, respectively, as $r \to \infty$. If ω is not purely imaginary, then substitution of the expression (6.23) into the field equations (6.17), retaining only terms up to first order in ε, and requiring that the coefficients of the linearly independent terms, $e^{i\omega t}$ and $e^{-i\omega^* t}$, separately vanish, gives

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}^1) + i\omega\, \vec{\nabla}\phi^1 - (\omega^2+2g^2)\vec{A}^1 - 4gg^1\vec{A} = 0 \tag{6.24.1}$$

$$\nabla^2\phi^1 + i\omega\, \vec{\nabla}\cdot\vec{A}^1 + 2g^2\phi^1 + 4\phi g g^1 = 0 \tag{6.24.2}$$

$$\nabla^2 g^1 + (\omega^2 + 2(\vec{A}^2 - \phi^2) - \frac{dN}{dg})g^1 + 4g(\vec{A}\cdot\vec{A}^1 - \phi\phi^1) = 0 \quad , \tag{6.24.3}$$

as well as the complex conjugate of these equations. If ω is purely imaginary, then $e^{i\omega t} = e^{-i\omega^* t}$ and we get only one set of equations, namely, the real part of the equations (6.24). The imaginary parts of $\vec{A}^1$, ϕ^1, and g^1 are arbitrary in this case.

If a sum of terms, corresponding to different frequencies, ω, had been included in the expressions (6.23), then the equations for the amplitude of the perturbation of one frequency, decouple from those for another. Therefore, even when more than one frequency component is present, we need only consider equations of the form (6.24). Attempts made to

find exact solutions to (6.24) using the exact solutions, (3.26), for ϕ and g, with $\vec{A}=0$, were not successful.

Now look more closely at the equations (6.24). Multiplying (6.24.1) by $\vec{A}^1$, adding ϕ^1 times (6.24.2), and subtracting the result from g^1 times (6.24.3) gives:

$$\omega^2\left(g^1,\phi^1,\vec{A}^1\right) = \frac{\int \left[(\vec{\nabla}\times\vec{A}^1)^2 - (\vec{\nabla}\phi^1)^2 + (\vec{\nabla}g^1)^2 + \left(\frac{dN}{dg}+2(\phi^2-\vec{A}^2)\right)(g^1)^2 + 2g^2\left((\phi^1)^2-(\vec{A}^1)^2\right) + 8gg^1(\phi\phi^1-\vec{A}\cdot\vec{A}^1)\right] d^3x}{\int \left((\vec{A}^1)^2+(g^1)^2 \right) d^3x} \tag{6.25}$$

The linear term in ω has dropped out because it can be converted to a surface integral at spatial infinity. We have assumed that the perturbations die off fast enough as $r\to\infty$ to make this integral vanish.

Since the equations (6.24) apply only when ω is not purely imaginary, (6.25) also has this restriction. When ω is purely imaginary, the appropriate equations to be satisfied are the real parts of (6.24). The integral expression for this negative ω^2 is exactly of the form (6.25), but with g^1, ϕ^1, and $\vec{A}^1$ replaced by their real parts, $\mathrm{Re}(g^1)$, $\mathrm{Re}(\phi^1)$, and $\mathrm{Re}(\vec{A}^1)$, respectively. Now compare this result to the expression (6.20) for δ^2E. If we let the real functions g', ϕ' and $\vec{A}'$, appearing in the integral for δ^2E, equal the real parts of g^1, ϕ^1, and $\vec{A}^1$, respectively, then we see that when ω is purely imaginary, the following equality holds:

$$\omega^2 = \frac{8\pi\,\delta^2E}{\int\left((\vec{A}')^2+(g')^2\right)d^3x} \tag{6.26}$$

Since $\vec{A}'$ and g' are real functions, (6.26) shows that ω^2 and δ^2E have the same sign, (negative), when ω is purely imaginary. When ω is not purely imaginary, a similar equality holds:

$$\omega^2 = \frac{8\pi\ \delta^2 E}{\int \left((\vec{A}^1)^2 + (g^1)^2\right) d^3x} \quad , \tag{6.27}$$

for all g^1, ϕ^1, and $\vec{A}^1$ satisfying the equations (6.24). However, as will be shown shortly, in this case g^1, ϕ^1, and $\vec{A}^1$ are not real functions so that the corresponding $\delta^2 E$ is not a real static variation of the type previously supposed. Also, since $\vec{A}^1$ and g^1 are not real, we cannot guarantee from (6.27), that $\delta^2 E$ and ω^2 have the same sign.

Assuming that g^1, ϕ^1, and $\vec{A}^1$ are real in (6.24) leads to the conditions

$$\begin{aligned} \mu\ (2\nu\vec{A}^1 - \vec{\nabla}\phi^1) &= 0 \\ \mu\ \vec{\nabla}\cdot\vec{A}^1 &= 0 \\ \mu\ \nu\ g^1 &= 0 \quad , \end{aligned} \tag{6.28}$$

where $\mu = \mathrm{Re}(\omega)$ and $\nu = \mathrm{Im}(\omega)$. If $\mu \neq 0$, then we find from (6.28) that either $\phi^1 = g^1 = 0$ and $\vec{A}^1 = 0$, or $\nu = 0$ and $\vec{\nabla}\phi^1 = 0$. The latter alternative is not consistent with our requirement that the perturbations vanish at spatial infinity. Therefore, there is no acceptable non-zero solution to (6.24) if g^1, ϕ^1, and A^1 are real and ω is not purely imaginary. So, under these conditions we are not able to relate ω^2 to $\delta^2 E$ for a real static variation.

Similar to the result for the scalar field example, (6.14), the relations (6.26) and (6.27) between $\delta^2 E$ and ω^2 hold only when the perturbations g^1, ϕ^1, and $\vec{A}^1$ satisfy the appropriate differential equations, (6.24). However, we have not been able to solve (6.24). To do practical calculations to determine the sign of $\delta^2 E$ or ω^2, it would be convenient to be able to use a variational trial function, such as a scale varied solution as was done for the scalar field, ψ, example. This method could be used in the latter case because the variational principle

$\delta\omega^2 = 0$, given in (6.13), is an extremum having a minimum character. We will show that the expression (6.25) for ω^2 is an extremum with respect to a certain type of variation. However, the appearance of derivative terms in the numerator with both signs suggests that it will be difficult to determine the maximum-minimum character of this extremum. A further problem is that the relation (6.25) is only valid if ω is not purely imaginary and as indicated previously, in this case the functions g^1, ϕ^1, and $\vec{A}^1$ cannot all be real.

To show that ω^2, given by (6.25), has some extremum properties, consider variations of the form

$$
\begin{aligned}
g^1 &\to g^1 + \varepsilon_1 g^{(2)} \\
\phi^1 &\to \phi^1 + \varepsilon_2 \phi^{(2)} \\
\vec{A}^1 &\to \vec{A}^1 + \varepsilon_3 \vec{A}^{(2)} \quad ,
\end{aligned}
\tag{6.29}
$$

where g^1, ϕ^1, and $\vec{A}^1$ are exact solutions of the equations (6.24) and $g^{(2)}$, $\phi^{(2)}$, and $\vec{A}^{(2)}$ are regular functions that vanish suitably fast at spatial infinity. Substitute (6.29) into the expression (6.25). For ease of notation, let

$$
\Delta \equiv \int \left((\vec{A}^1)^2 + (g^1)^2 \right) d^3x \quad . \tag{6.30}
$$

Then one finds:

$$
\begin{aligned}
\delta_1\omega^2 &\equiv \left. \frac{\partial \omega^2}{\partial \varepsilon_1} \right|_{\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0} \\
&= \frac{1}{\Delta} \int \left[2 \vec{\nabla} g^1 \cdot \vec{\nabla} g^{(2)} + 2 \left(\frac{dN}{dg} + 2(\phi^2 - \vec{A}^2) \right) g^1 g^{(2)} + 8 g g^{(2)} (\phi \phi^1 - \vec{A} \cdot \vec{A}^1) \right] d^3x \\
&\qquad - \frac{\omega^2}{\Delta} \int 2 g^1 g^{(2)} \, d^3x \quad .
\end{aligned}
\tag{6.31}
$$

Integrating by parts in the first term in (6.31) and dropping the surface term gives:

$$(6.32)\quad \delta_1\omega^2 = -\frac{2}{\Delta}\int g^{(2)}\left[\ \nabla^2 g^1 + \left(\omega^2+2(\vec{A}^2-\phi^2)-\frac{dN}{dg}\right)g^1 + 4g(\vec{A}\cdot\vec{A}^1-\phi\phi^1)\ \right] d^3x$$

$$= 0 \quad , \text{ using (6.24.3)} \quad .$$

Similarily,

$$(6.33)\qquad \delta_2\omega^2 \equiv \left.\frac{\partial\omega^2}{\partial\varepsilon_2}\right|_{\varepsilon_1=\varepsilon_2=\varepsilon_3=0}$$

$$= \frac{1}{\Delta}\int\left[\ -2\vec{\nabla}\phi^1\cdot\vec{\nabla}\phi^{(2)} + 4g^2\phi^1\phi^{(2)} + 8gg^1\phi\phi^{(2)}\ \right] d^3x$$

$$= \frac{2}{\Delta}\int\left[\nabla^2\phi^1+2g^2\phi^1+4\phi gg^1\ \right]\phi^{(2)}\ d^3x \quad .$$

Substituting in (6.32) from the equation (6.24.2) gives:

$$(6.34)\qquad \delta_2\omega^2 = -\frac{2i\omega}{\Delta}\int \phi^{(2)}\vec{\nabla}\cdot\vec{A}^1\ d^3x \quad .$$

Finally,

$$(6.35)\qquad \delta_3\omega^2 \equiv \left.\frac{\partial\omega^2}{\partial\varepsilon_3}\right|_{\varepsilon_1=\varepsilon_2=\varepsilon_3=0}$$

$$= \frac{1}{\Delta}\int\left[\ 2(\vec{\nabla}\times\vec{A}^1)\cdot(\vec{\nabla}\times\vec{A}^{(2)}) - 4g^2\vec{A}^1\cdot\vec{A}^{(2)} - 8gg^1\vec{A}\cdot\vec{A}^{(2)}\ \right] d^3x$$

$$-\frac{\omega^2}{\Delta}\int 2\vec{A}^1\vec{A}^{(2)}\ d^3x$$

$$= \frac{2}{\Delta}\int\left[\ \vec{\nabla}\times(\vec{\nabla}\times\vec{A}^1)-(\omega^2+2g^2)\vec{A}^1-4gg^1\vec{A}\ \right]\cdot\vec{A}^{(2)}\ d^3x \quad .$$

Substituting from (6.24.3) leaves

$$(6.36)\qquad \delta_3\omega^2 = -\frac{2i\omega}{\Delta}\int\vec{\nabla}\phi^1\cdot\vec{A}^{(2)}\ d^3x \ = \ \frac{2i\omega}{\Delta}\int\phi^1\vec{\nabla}\cdot\vec{A}^{(2)}\ d^3x \quad .$$

Therefore, expanding $\omega^2(\varepsilon_1,\varepsilon_2,\varepsilon_3,)$ in a Taylor series in ε_1, ε_2, and ε_3,

we have from (6.32), (6.34) and (6.36):

$$\omega^2(\varepsilon_1,\varepsilon_2,\varepsilon_3) = \omega^2\Big|_{\varepsilon_1=\varepsilon_2=\varepsilon_3=0} - \frac{2i\omega}{\Delta}\int\left(\varepsilon_2\phi^{(2)}\vec{\nabla}\cdot\vec{A}^1-\varepsilon_3\phi^1\vec{\nabla}\cdot\vec{A}^{(2)}\right)d^3x + \text{terms of higher order.} \tag{6.37}$$

Therefore, if $\vec{\nabla}\cdot\vec{A}^1=0$, then ω^2 given by (6.25) is an extremum with respect to the restricted class of variations about the exact solutions g^1, ϕ^1, and $\vec{A}^1$, such that $\vec{\nabla}\cdot\vec{A}^{(2)}=0$, because in this case, the first order terms in ε_1, ε_2, and ε_3 all vanish in (6.37).

This need to impose a gauge condition such as $\vec{\nabla}\cdot\vec{A}=0$, in order to make ω^2 an extremum, arises from the linear terms in ω in (6.24), which in turn, come from the first order time derivatives in the original field equations. The expression (6.25) is not a 'Lagrangian' for the system (6.24). That is, requiring $\delta\omega^2=0$ for arbitrary variations does not generate the equations (6.24). This is to be compared to the example involving the scalar field ψ, for which only quadratic terms in ω appear and the expression (6.13) for ω^2 is a 'Lagrangian' for the equation (6.12).

The explicit appearance of linear terms in ω can be eliminated in the equations (6.24) if the fields $\tilde{\phi}$ and $\tilde{A}$ satisfy the Lorentz condition $\tilde{A}_{\mu,\mu}=0$. However, as our theory is not gauge invariant, we should consider other gauges as well. If one could show that a solution is unstable with respect to perturbations satisfying some additional gauge restriction, then we could conclude that the original system is unstable. However, if one finds stability with respect to perturbations in a special gauge, this would not show stability of the system with respect to unrestricted perturbations.

Although our analysis of the stability of solutions of the system (6.17) has not progressed very far, there are several things that have been learned. General relations, such as (6.27), exist between $\delta^2 E$ and ω^2 if one computes $\delta^2 E$ for complex, rather than only real, static variations. In the particular case that ω is purely imaginary, assuming solutions exist, ω^2 and $\delta^2 E$ have the same (negative) sign. We have seen that, in a simple case, $\delta^2 E$ evaluated for a scale varied solution, for example, can give information about stability through its relation to ω^2 resulting from a minimum variational principle. However, because of the complexity of the system involving g and A_μ, we have been unable to establish comparable results.

CHAPTER 7

SUMMARY

We have seen that there exist nonlinear field equations for the electromagnetic potential, A_μ, and a real scalar field, f, such that the charge is fixed for all solutions of a certain class. Specifically, there are systems for which all spherically symmetric solutions with power series expansions of the form (2.33), have a charge which is restricted to the values $-q$, 0, or q, for some number, q. However, charged particle solutions of this type are found to interact with one another according to a force which drops off more rapidly than the inverse square of their separation. Other systems, with a (possibly) larger spectrum of discrete charges, have the proper Coulomb force law for particles momentarily at rest. Furthermore, exact solutions were found to the field equations of this type, which are finite everywhere and have a finite, non-negative rest energy.

These results were derived in what is essentially a phenomenological approach to finding a nonlinear modification of the Maxwell equations suitable for describing extended particles. An investigation of a detailed model of the background radiation field, assumed to give rise to the scalar field, f, might delimit the possible form for the nonlinear contributions. A more fundamental theory might also lead to definite statements about the asymptotic boundary condition appropriate for the field f (or g), with important consequences for the determination of the allowed charge values. We regard the present work as an exploratory study, to give an indication of the potential of the concept of fixing the charge through the field equations. In this spirit, there are still

a number of questions to be examined. In addition to the possibility of looking at other forms for the Lagrangian, even for the systems of the type already considered, there remains the important question of whether or not the charge can be fixed for time-dependent solutions. One might also examine the relation between the charge fixing problem for angular dependent solutions and the simpler case of spherical symmetry.

While some of the systems we have studied have many desirable features, such as a discrete charge spectrum and finite, non-negative energy, there are improvements or generalizations to be sought in other areas. The angular momentum for all the exact solutions that we found, turned out to be zero. It would be interesting to find a classical field theory model in which there are particle-like solutions with non-zero total angular momentum. In this regard, one might also look for solutions which are periodic in time, rather than static.

Also concerning angular momentum, we have seen that there are problems associated with the division of the total angular momentum into spin and orbital components, in the case where the canonical energy-momentum tensor is not symmetric. These include the fact that the separation depends on the choice of hypersurface used in the definitions, and on the specific form of equivalent† Lagrangians. The observed proportionality of the spin and orbital angular momenta in some cases of interest, the translational invariance of the orbital angular momentum for localized solutions in a zero-momentum frame, and the failure to exhibit a general proof that the orbital angular momentum vanishes for such solutions, all cast doubt on the definitions used for the quantities involved.

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† Equivalent Lagrangians are those generating the same field equations.

If a zero momentum, localized solution was found, it would provide motivation for an attempt to find a new definition for orbital angular momentum in classical field theory. In any case, further study in this area seems indicated.

Finally, with regard to the stability of solutions in classical field theory, we were unable to make definite statements about the particular systems of interest in this paper. We were not able to conclude that the static criterion $\delta^2 E<0$, is equivalent to the dynamical criterion for instability, $\omega^2<0$, for these systems, although in certain cases the two tests can be shown to agree. Even if one could decide on the nature of the stability of a solution to a classical field theory, one still has to answer the question of whether or not stability should even be required at the classical level. There is much room for investigation in the whole area of stability.

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APPENDIX 1

FOUR-MOMENTUM IN THE TIME-INDEPENDENT CASE

A1.1 Definition of the Four-momentum

The invariance of the Lagrangian, for a closed system, under translations in space and time leads directly to the conservation of the four-momentum, P_μ.[†]

$$P_\mu \equiv \int_\sigma T_{\mu\nu}\, d\sigma_\nu \quad , \tag{A1.1}$$

is the integral of the normal component of the canonical energy-momentum tensor, $T_{\mu\nu}$, over a space-like surface, σ. The tensor, $T_{\mu\nu}$, is defined in terms of the Lagrangian density, $\mathcal{L} = \mathcal{L}(A_\mu, A_{\mu,\nu}, f, f_{,\nu})$ by:

$$T_{\mu\nu} \equiv \delta_{\mu\nu}\mathcal{L} - A_{\sigma,\mu}\frac{\partial \mathcal{L}}{\partial A_{\sigma,\nu}} - f_{,\mu}\frac{\partial \mathcal{L}}{\partial f_{,\nu}} \quad . \tag{A1.2}$$

If $\mathcal{L}$ does not depend explicitly on the coordinates x_μ, as is assumed here, then $T_{\mu\nu,\nu} = 0$, when evaluated for fields satisfying the field equations. This condition implies that P_μ is independent of the surface, σ. This is an expression of the conservation of the energy-momentum four-vector P_μ.

If we take σ to be a surface of constant time, then

$$P_\mu = -i \int T_{\mu 4}\, d^3x \quad . \tag{A1.3}$$

The space components, P_k, define the three-momentum, $\vec{P}$, and the energy, E, is given by

$$E = -i\, P_4 = -\int T_{44}\, d^3x \quad . \tag{A1.4}$$

[†] See, for example, Aharoni (1965), Barut (1964), or Rzewuski (1967).

A1.2 Three-momentum in the Static Case

In a reference frame in which the fields are independent of time, one can show that the components, P_k, of the three-momentum, vanish. We will call this a rest frame of the system.

To show that $P_k=0$ for time-independent systems, we introduce a symmetrical energy-momentum tensor $\Theta_{\mu\nu}$, which is known to exist for Lorentz invariant Lagrangians. The tensor $\Theta_{\mu\nu} = \Theta_{\nu\mu}$, is defined such that $\Theta_{\mu\nu,\nu} = 0$ and it gives the same momentum when integrated, as does the canonical tensor, $T_{\mu\nu}$:

$$P_k = -i \int T_{k4}\, d^3x = -i \int \Theta_{k4}\, d^3x = -i \int \Theta_{4k}\, d^3x \quad . \tag{A1.5}$$

But, this last integral is here shown to vanish under quite general circumstances. Using $\Theta_{\mu\nu,\nu}=0$ we have:

$$\begin{aligned} \int \Theta_{\mu k}\, d^3x &= \int (x_k \Theta_{\mu\nu})_{,\nu}\, d^3x \\ &= \int (x_k \Theta_{\mu j})_{,j}\, d^3x \quad , \text{ in the static case.} \\ &= \int_S x_k \Theta_{\mu j}\, dS_j \quad , \text{ where S is a surface at } r=\infty \\ &= 0 \quad , \text{ if } \Theta_{\mu j} \text{ falls off faster than } r^{-3} \text{ as } r\to\infty. \end{aligned} \tag{A1.6}$$

So, $P_k = 0$.

Therefore we can always find a rest frame for the system, if there is a time-independent solution in which the energy-momentum tensor falls off faster than r^{-3} as $r \to \infty$. (A1.6) also shows that the self stress, $\int \Theta_{jk}\, d^3x$, vanishes under these conditions.

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† $\Theta_{\mu\nu}$ was originally introduced by Belinfante (1940) and Rosenfeld (1940), but one could also see any of the books in the previous footnote.

A1.3 Rest Energy

We now look at the energy, $E = -\int T_{44}\, d^3x$. In the case of a time-independent system, E is the rest energy and is given by

$$E = -\int \mathcal{L}\, d^3x \ .$$

Look at the particular example considered in Chapter 2. In the static case, with the vector potential, $\vec{A} = 0$, the Lagrangian density, $\mathcal{L}$, is given by (2.16)

$$(2.16) \qquad \mathcal{L} = -\frac{1}{4\pi}\left(\frac{1}{2} B\, (\vec{\nabla} g)^2 - \frac{1}{2} D\, (\vec{\nabla}\phi)^2 + G(\phi, g) \right) ,$$

where B and D are constants. The fact that the field equations we consider are derived from a Lagrangian through a variational principle, allows one to obtain integral relations satisfied by solutions of the equations. In particular, the static Lagrangian, $L = \int \mathcal{L}\, d^3x = -E$, is an extremum for a solution of the field equations, compared to functions obtained from this solution by scale or amplitude variations. In the expression (2.16) replace $\phi(\vec{r})$ by $\nu\phi(\lambda\vec{r})$, and $g(\vec{r})$ by $\mu g(\lambda\vec{r})$, with $\lambda>0$, and calculate the Lagrangian

$$(A1.7) \quad L = L(\mu,\nu,\lambda) = -\frac{1}{4\pi}\int\left(\frac{B}{2}\mu^2(\vec{\nabla} g(\lambda\vec{r}))^2 - \frac{D}{2}\nu^2(\vec{\nabla}\phi(\lambda\vec{r}))^2 + G(\mu g(\lambda\vec{r}), \nu\phi(\lambda\vec{r}))\right) d^3x \ .$$

Assuming that $\phi(\vec{r})$ and $g(\vec{r})$ are solutions of the field equations, the extremum conditions mentioned above take the form

$$(A1.8) \qquad \frac{\partial L}{\partial \mu} = \frac{\partial L}{\partial \nu} = \frac{\partial L}{\partial \lambda} = 0 \ ,$$

where all the derivatives are evaluated at $\mu=\nu=\lambda=1$.

Carrying out the differentiations in (A1.8) gives the following

integral relations:

$$\int \left(- D \, (\vec{\nabla}\phi)^2 + \phi \, \frac{\partial G}{\partial \phi} \right) d^3x = 0$$

$$\int \left(B \, (\vec{\nabla} g)^2 + g \, \frac{\partial G}{\partial g} \right) d^3x = 0 \qquad \text{(A1.9)}$$

$$\int \left(B \, (\vec{\nabla} g)^2 - D \, (\vec{\nabla}\phi)^2 + 6 \, G \right) d^3x = 0 \; .$$

Taking a general linear combination of the integrals in (A1.9), one finds that the rest energy, $E = -L$, can be expressed as

$$\text{(A1.10)} \qquad E = \frac{1}{4\pi} \int \left[\left(\tfrac{1}{2}+\beta+\tfrac{\gamma}{2}\right) B \, (\vec{\nabla} g)^2 - \left(\tfrac{1}{2}+\alpha+\tfrac{\gamma}{2}\right) D \, (\vec{\nabla}\phi)^2 + \alpha \, \phi \, \frac{\partial G}{\partial \phi} + \beta \, g \, \frac{\partial G}{\partial g} + (1+3\gamma) \, G \right] d^3x \; .$$

Here, α, β, and γ, are arbitrary constant parameters. In the particular case where $\alpha = \beta = -\frac{1}{2}(1+\gamma)$, we get

$$\text{(A1.11)} \qquad E = \frac{1}{4\pi} \int \left[-\frac{1}{2} (1+\gamma) \left(\phi \, \frac{\partial G}{\partial \phi} + g \, \frac{\partial G}{\partial g} \right) + (1+3\gamma) \, G \right] d^3x \; .$$

Choosing $\gamma = -1$ gives the particularily simple expression

$$\text{(A1.12)} \qquad E = -\frac{1}{2\pi} \int G \, d^3x \; .$$

In Chapter 3 we studied systems based on the Lagrangian density

$$\text{(A1.13)} \qquad = -\frac{1}{4\pi} \left[\frac{1}{2} \, (\vec{\nabla}\times\vec{A})^2 - \frac{1}{2} \, (\vec{\nabla}\phi)^2 + \frac{1}{2} \, (\vec{\nabla} g)^2 + \frac{\delta}{2} \, (\vec{\nabla}\cdot\vec{A})^2 + (\phi^2 - \vec{A}^2) \, g^2 + K(g) \, \vec{\nabla}\cdot\vec{A} + M(g) \right] .$$

Using the technique of deriving integral relations from scale and amplitude variations, one finds in this case that the energy of solutions of the field equations can be written as

$$(A1.14)\quad E = -\int \mathcal{L}\, d^3x$$

$$= \frac{1}{4\pi}\int \Big\{ \Big(\tfrac{1}{2}(\vec{\nabla}\times\vec{A})^2 + \tfrac{\delta}{2}(\vec{\nabla}\cdot\vec{A}) \Big)\,(1+2\gamma+\lambda) - \tfrac{1}{2}(\vec{\nabla}\phi)^2\,(1+2\alpha+\lambda)$$

$$+ \tfrac{1}{2}(\vec{\nabla}g)^2\,(1+2\beta+\lambda) + K\,\vec{\nabla}\cdot\vec{A}\,(1+\gamma+2\lambda) + \beta\, g\,\frac{dK}{dg}\,\vec{\nabla}\cdot\vec{A}$$

$$+ \phi^2 g^2\,(1+2\alpha+2\beta+3\lambda) - \vec{A}^2 g^2\,(1+2\beta+2\gamma+3\lambda)$$

$$+ M\,(1+3\lambda) + \beta\, g\,\frac{dM}{dg}\Big\}\, d^3x \quad .$$

Here, α, β, γ, and λ are arbitrary constants. With the particular choice $\beta = 0$ and $\alpha = \gamma = \lambda = -\frac{1}{3}$, we find from (A1.14)

$$(A1.15)\qquad E = \frac{1}{12\pi}\int \Big\{ (\vec{\nabla}g)^2 + 2\,(\vec{A}^2 - \phi^2)\, g^2 \Big\}\, d^3x \quad .$$

APPENDIX 2

DERIVATION OF CONDITIONS TO FIX THE CHARGE

In this appendix we derive some of the conditions which may be required for the Lagrangian

$$(2.16) \qquad \mathcal{L} = -\frac{1}{4\pi}\left[\frac{1}{2} B (\vec{\nabla} g)^2 - \frac{1}{2} D (\vec{\nabla}\phi)^2 + G(\phi, g)\right]$$

to lead to a fixed charge system. We assume that $G(\phi,g)$ is given by

$$(2.20) \qquad G(\phi,g) = \sum_{m,n} G_{mn} \phi^m g^n \ .$$

We look for solutions of the form

$$(2.34) \qquad \phi = \sum_{n=1}^{\infty} \frac{b_n}{r^n}$$
$$g = \sum_{n=1}^{\infty} \frac{g_n}{r^n}$$

to the field equations

$$(A2.1) \qquad g = \frac{1}{B}\frac{\partial G}{\partial g} = \sum_{m,n} \alpha_{mn} \phi^m g^n$$

$$(A2.2) \qquad \phi = -\frac{1}{D}\frac{\partial G}{\partial \phi} = \sum_{m,n} \beta_{mn} \phi^m g^n \ .$$

Substitution of the series (2.34) into the field equations (A2.1) and (A2.2) and equating the coefficients of like powers of r on each side gives the following conditions; (P_m, Q_m, S_m, and T_m are defined by equations (2.36) through (2.39).)

$$(A2.3.0) \qquad 0 = P_0 = \alpha_{00}$$

$$(A2.3.1) \qquad 0 = P_1 = \alpha_{10} + \varepsilon S_1$$

(A2.3.2) $$0 = P_2 b_1^2 + S_1(g_2 - \varepsilon b_2)$$

(A2.3.3) $$0 = P_3 b_1^3 + 2P_2 b_1 b_2 + S_2 b_1 (g_2 - \varepsilon b_2) + S_1(g_3 - \varepsilon b_3)$$

(A2.3.4) $$2g_2 = P_4 b_1^4 + 3P_3 b_1^2 b_2 + 2P_2 b_1 b_3 + S_3 b_1^2 (g_2 - \varepsilon b_2)$$
$$+ S_2 b_1 (g_3 - \varepsilon b_3) + S_1(g_4 - \varepsilon b_4) + \alpha_{20} b_2^2 + \alpha_{11} b_2 g_2 + \alpha_{02} g_2^2$$

(A2.3.n) $$\tfrac{1}{2}n(n-1)\, g_{n-2} = P_n b_1^n + nP_{n-1} b_1^{n-2} b_2 + \ldots + S_1(g_n - \varepsilon b_n) + \ldots$$

and,

(A2.4.0) $$0 = Q_0$$

(A2.4.1) $$0 = Q_1 = \beta_{10} + \varepsilon T_1$$

(A2.4.2) $$0 = Q_2 b_1^2 + T_1(g_2 - \varepsilon b_2)$$

(A2.4.3) $$0 = Q_3 b_1^3 + 2Q_2 b_1 b_2 + T_2 b_1 (g_2 - \varepsilon b_2) + T_1(g_3 - \varepsilon b_3)$$

(A2.4.n) $$\tfrac{1}{2}n(n-1)\, b_{n-2} = Q_n b_1^n + nQ_{n-1} b_1^{n-2} b_2 + \ldots + T_1(g_n - \varepsilon b_n) + \ldots$$

Here, equations (A2.3.n) and (A2.4.n) are obtained from the r^{-n} term in the g and ϕ equations, respectively.

We want the solutions of these field equations to have finite energy. From Appendix 1, the energy may be expressed as

(A1.12) $$E = -\frac{1}{2\pi} \int G \, d^3x \quad .$$

If G goes to zero slower than r^{-4}, as $r \to \infty$, the energy will not be finite. To examine the asymptotic properties of G, substitute the expansions, (2.34) for ϕ and g into the expression (2.20) for G. One finds

$$G = G_{00} + \frac{1}{r}\left(\varepsilon BP_0 - DQ_0\right)b_1 + \frac{1}{2r^2}\left(\varepsilon BP_1 - DQ_1\right)b_1^2 + \frac{1}{3r^3}\left(\varepsilon BP_2 - DQ_2\right)b_1^3 + O(r^{-4}).$$

If $G_{00} = 0$, then one finds, using the field equations, that all terms up to and including r^{-3}, are identically zero, without placing any further constraints on the parameters appearing in the Lagrangian. Therefore, the only condition we get from the requirement of finite energy, is $G_{00} = 0$.

We now show that the field equations can fix the charge, if the Lagrangian is chosen properly. Certain conditions must be satisfied by the parameters in the Lagrangian, if solutions with non-zero charge, b_1, are to exist, and in particular, if this charge is to have a fixed magnitude. As a starting point, in this section, we assume that the parameter, $T_1 = \beta_{01} \neq 0$. Also, throughout, it is assumed that $b_1 \neq 0$.

Equations (A2.3.0) and (A2.4.0) require the vanishing of the coefficients α_{00} and β_{00}, the terms in the field equations which are independent of ϕ and g. Equation (A2.3.1) serves to determine the value

$$\varepsilon \equiv \frac{g_1}{b_1} = -\frac{\beta_{10}}{T_1} \quad . \tag{A2.5}$$

Consistency of equations (A2.3.1) and (A2.4.1) requires

$$\alpha_{10}T_1 = \beta_{10}S_1 \quad . \tag{A2.6}$$

Eliminating $g_2 - \varepsilon b_2$ from (A2.3.2) and (A2.4.2); and $g_3 - \varepsilon b_3$ from (A2.3.3) and (A2.4.3) gives:

$$T_1P_2 = S_1Q_2 \quad , \text{ and} \tag{A2.7}$$

(A2.8) $$T_1P_3 + T_2P_2 = S_1Q_3 + S_2Q_2 \quad .$$

The requirements listed thus far are necessary for the existence of solutions with non-zero charge, but the question of fixing the charge does not arise until one considers equations (A2.3.4) and (A2.4.4). Eliminating $g_4-\varepsilon b_4$ from these equations gives

(A2.9) $$-2\,(S_1-\varepsilon T_1)b_2+Q_2b_1^2 = \left[(T_1P_4-S_1Q_4)+(T_2P_3-S_2Q_3)+(T_3P_2-S_3Q_2)+(T_1\alpha_{02}-S_1\beta_{02})\frac{Q_2^2}{T_1^2}\right]b_1^4$$

If the coefficient of b_1 is non-zero, and if $S_1-\varepsilon T_1 = 0$, the above expression involves only b_1, and therefore fixes the charge. If $S_1-\varepsilon T_1 \neq 0$, the equations (A2.3.4) and (A2.4.4) do not fix the charge and the higher order equations (A2.3.5) and (A2.4.5) must be considered. However, it was found that the charge did not appear to be fixed by these higher equations unless $S_1-\varepsilon T_1 = 0$.

In addition to the constraints (A2.6), (A2.7), and (A2.8) already mentioned, the parameters are subject to the condition

(2.40) $$mBP_m = -DT_m + \varepsilon BS_m \quad ,$$

that the field equations be derivable from a Lagrangian. When $m = 1$, this relation, combined with equations (A2.3.1) and (A2.4.1), and the condition $S_1-\varepsilon T_1 = 0$, gives the result

(A2.10) $$D = \varepsilon^2 B \quad .$$

To summarize, if the conditions (A2.6), (A2.7), (A2.8), and (2.40) are satisfied, with $S_1 = \varepsilon T_1$, and ε determined by (A2.5), then all charged

solutions, if any exist, with a power series expansion of the assumed form, have the same magnitude of charge, $q = |b_1|$, where

$$(A2.11) \qquad q^2 = \frac{-2Q_2}{(T_1P_4-S_1Q_4)+(T_2P_3-S_2Q_3)+(T_3P_2-S_3Q_2)+(T_1\alpha_{02}-S_1\beta_{02})\frac{Q_2^2}{T_1^2}}$$

To fix the charge at a non-zero value, using equations (A2.3.4) and (A2.4.4), the parameter Q_2 must be non-zero. If $Q_2 = 0$, one finds that it is still possible to fix the charge by using equations (A2.3,5) and (A2.4.5). In this case, the necessary conditions are:

$$(A2.12) \qquad P_0=P_1=P_2=0=Q_0=Q_1=Q_2 \; ; \; P_3=\varepsilon Q_3 \; ; \; P_4=\varepsilon Q_4 \; ; \; S_1=\varepsilon T_1 \; ; \text{ and } S_2=\varepsilon T_2 \; .$$

The value of the fixed charge is q, where

$$(A2.13) \qquad q^2 = \frac{-6Q_3}{T_1(P_5-\varepsilon Q_5) - 3Q_3^2} \; .$$

Investigation of the case $Q_3 = 0$, did not reveal any further systems in which all the spherically symmetric solutions, of the assumed form, have the same charge.

APPENDIX 3

IMPOSING THE CONDITION $\vec{\nabla}\cdot\vec{A} = 0$

Given the expression (3.29), reproduced below, for the Cartesian components, A_i, of the vector potential, we want to impose the supplementary condition (3.28): $\vec{\nabla}\cdot\vec{A} = A_{i,i} = 0$. Here A_i is given by

$$(3.29) \qquad A_i(\vec{r}) = \sum_{l=0}^{n-1} \sum_{m=-l}^{l} (k_i)_{lm}\, Y_{lm}(\theta,\phi)\, \phi_{nl}(r) \quad ; \quad n=1,2,3 \ldots \ .$$

For small values of n, when there are few terms in the sum (3.29), it is relatively easy to calculate $A_{i,i}$, but for larger n the problem is much more difficult. An alternative way to proceed is based on the equation satisfied by the function (3.29), namely

$$(3.27) \qquad \nabla^2\vec{A} = -2g^2\vec{A} \ .$$

Taking the divergence of (3.27), using the vector identity

$$\nabla^2\vec{A} - \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = -\vec{\nabla}\times(\vec{\nabla}\times\vec{A})$$

and the fact that the divergence of a curl vanishes, one finds

$$(A3.1) \qquad 0 = \nabla^2(\vec{\nabla}\cdot\vec{A}) + 2g^2(\vec{\nabla}\cdot\vec{A}) + 4g\vec{A}\cdot\vec{\nabla}g \ .$$

This shows, that for solutions in which $\vec{\nabla}\cdot\vec{A}=0$, and $g=g(r)\neq 0$, the radial component, A_r, of the vector potential, vanishes.

When n=1, so only the l=0 term contributes to (3.29), one finds the condition, $A_r=0$, can be satisfied at all points in space only when $\vec{A}=0$. When n=2, a non-zero result is obtained, of the form

$$(A3.2) \qquad \vec{A} = \frac{\vec{k}\times\vec{r}}{R^3} = \frac{\vec{k}\times\vec{r}}{(r^2+a^2)^{\frac{3}{2}}} \ ,$$

where $\vec{k}$ is an arbitrary real constant vector. The form (A3.2) is of particular interest because in the limit $a \to 0$, or $r \to \infty$, this is exactly the form of the vector potential of a particle with a magnetic dipole moment, $\vec{k}$. For all $n \geq 2$, solutions exist with the dipole structure, characterized by the $\vec{k}\times\vec{r}$ factor and the $\frac{1}{r^2}$ asymtotic ($r \to \infty$) behaviour. Specifically,

$$\vec{A} = \frac{\vec{k}\times\vec{r}}{R^3} F(n+2, -n+2; \frac{5}{2}; \frac{a^2}{R^2})$$

is a solution. Note this includes contributions only from the $l=1$ terms in the expansion (3.29), so we expect that there exist more general solutions. This is found to be the case.

When $n=3$, the condition $A_r=0$ no longer is sufficient to ensure that the constraint $\vec{\nabla}\cdot\vec{A}=0$, is satisfied. The form of the solution for $n=2$ suggests a different way to proceed. Choosing a coordinate system in which $\vec{k}$ is along the z axis, the expression (A3.2) is of the form $\vec{A}(\vec{r}) = A(r)\ \vec{l}_\phi$, where $\vec{l}_\phi$ is a unit vector in the direction of increasing ϕ. To generalize this result, we look for a solution of the form

(A3.3) $$\vec{A}(\vec{r}) = A(r,\theta)\ \vec{l}_\phi \ ,$$

for which the condition $\vec{\nabla}\cdot\vec{A}=0$ is identically satisfied.

Separation of variables in the equation

(3.7.1) $$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = 2g^2\vec{A}$$

can be used to find a solution of the form (A3.3). Let $A(r,\theta) = X(r)Y(\theta)$. Writing the separation constant as $l(l+1)$ for convenience, one finds that $X(r)$ and $Y(\theta)$ satisfy the equations

(A3.4) $$\frac{1}{r}\frac{d^2}{dr^2}(rX) + \left(\frac{2c_n^2}{R^2} - \frac{l(l+1)}{r^2}\right) X = 0$$

and

$$\text{(A3.5)} \qquad \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dY}{d\theta}\right) + \left(1(1+1) - \frac{1}{\sin^2\theta}\right) Y = 0 \quad ,$$

respectively. Here, we have used the expression $g = \frac{c_n}{R^2} \equiv \frac{a\left[\frac{4n^2-1}{2}\right]^{\frac{1}{2}}}{a^2+r^2}$. The equation (A3.4) for X has already been solved because it is identical in structure to (3.11). It has the regular solutions

$$\text{(A3.6)} \qquad X(r) = X_l(r) = \phi_{nl} \quad ; \quad l=0,1,2, \ldots, n-1 \quad ; \quad n=0,1,2, \ldots \quad ,$$

with ϕ_{nl} given by (3.20).

In treating the Y equation, (A3.5), it was found that the substitution $Y(\theta) = \sin\theta\, W(\theta)$ reduces the equation to a hypergeometric equation of the form (3.14) for W, in the independent variable $\sin^2\theta$. This leads to the solution

$$\text{(A3.7)} \qquad Y(\theta) = \sin\theta\, F\left(l+\tfrac{1}{2}, \tfrac{1}{2}-\tfrac{1}{2}; 2; \sin^2\theta\right) \quad ; \quad l=1,2,3, \ldots \quad .$$

For l=0, a solution exists of the form

$$Y(\theta) = \frac{\sin\theta}{1\pm\cos\theta} \quad ,$$

but we exclude this case because this solution is not continuous in the closed interval $(0,\pi)$. $Y(\theta)$ in (A3.7) can also be expressed in terms of Legendre polynomials †

$$\text{(A3.8)} \qquad Y(\theta) = \frac{2\sin\theta}{1(1+1)}\frac{d}{d(\cos\theta)}P_l(\cos\theta) \quad ; \quad l=1,2, \ldots \quad .$$

Combining the results (A3.6) and (A3.7) for X and Y respectively, one finds that $\vec{A} = A(r,\theta)\,\vec{I}_\phi$ is a solution to the equation (3.7.1), with $g = \frac{c_n}{R^2}$, if $A(r,\theta)$ is given by

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† Erdélyi et.al. (1953), p 147

$$(A3.9) \qquad A(r,\theta) = \sum_{l=1}^{n-1} \kappa_l \frac{r^l}{R^{2l+1}} F(l+n+1, l-n+1; l+\frac{3}{2}; \frac{a^2}{R^2}) \times \sin\theta \, F(l+\frac{1}{2}, \frac{1}{2}-\frac{1}{2}; 2; \sin^2\theta) \quad ,$$

where κ_l are constants that may depend on n. n can take on the values n=2,3,4, ...

Alternatively, in terms of orthogonal polynomials, from (3.22) and (A3.8), we find

$$(A3.10) \qquad A(r,\theta) = \sum_{l=1}^{n-1} \kappa'_l \frac{r^l}{R^{2l+1}} C_{n-l-1}^{l+1}\left(\frac{r^2-a^2}{r^2+a^2}\right) \sin\theta \frac{d}{d(\cos\theta)} P_l(\cos\theta) \quad ,$$

where

$$\kappa'_{nl} = \frac{2}{l(l+1)} \frac{1}{\binom{n+1}{2l+1}} \kappa_{nl} \quad .$$

It should be noted that the preceding expressions may not be the general solution to the equation (3.7.1) for $\vec{A}$. For n>2 it may even be possible to find a solution satisfying $\vec{\nabla}\cdot\vec{A}=0$ in which $\vec{A}$ is not of the form (A3.3).

APPENDIX 4

RELATION BETWEEN ORBITAL AND SPIN ANGULAR MOMENTA

We want to calculate the spin and orbital angular momenta for systems described by Lagrangians of the form

$$\mathcal{L} = U\mu + V\nu + \bar{\mathcal{L}}(\omega,\chi,\xi,\zeta,g), \tag{5.39}$$

where U and V are constants and μ,ν,ω,χ,ξ, and ζ are the scalar functions of the fields A_α and g defined by (5.38). From (5.38) and (5.39) one can calculate

$$\frac{\partial \mathcal{L}}{\partial A_{\alpha,\beta}} = U A_{\beta,\alpha} + V A_{\alpha,\beta} + \frac{\partial \mathcal{L}}{\partial \omega}\delta_{\alpha\beta} \tag{A4.1}$$

$$\frac{\partial \mathcal{L}}{\partial g_{,\alpha}} = \frac{\partial \mathcal{L}}{\partial \chi} g_{,\alpha} + \frac{\partial \mathcal{L}}{\partial \xi} A_\alpha \ . \tag{A4.2}$$

Using (A4.1) and (A4.2), one finds the field equations derived from the Lagrangian (5.39) are

$$\begin{aligned} U A_{\beta,\alpha\beta} + V A_{\alpha,\beta\beta} + \left(\frac{\partial \mathcal{L}}{\partial \omega}\right)_{,\alpha} &= \frac{\partial \mathcal{L}}{\partial \xi} g_{,\alpha} + 2\frac{\partial \mathcal{L}}{\partial \zeta} A_\alpha \\ \left(\frac{\partial \mathcal{L}}{\partial \chi} g_{,\alpha} + \frac{\partial \mathcal{L}}{\partial \xi} A_\alpha\right)_{,\alpha} &= \frac{\partial \mathcal{L}}{\partial g} \ . \end{aligned} \tag{A4.3}$$

The other quantity we will need in the calculation of the angular momenta is the canonical energy-momentum tensor

$$T_{\mu\nu} \equiv \delta_{\mu\nu}\mathcal{L} - A_{\sigma,\mu}\frac{\partial \mathcal{L}}{\partial A_{\sigma,\nu}} - g_{,\mu}\frac{\partial \mathcal{L}}{\partial g_{,\nu}} \ . \tag{5.10}$$

In particular, for a time-independent case, we have

$$\begin{aligned} T_{m4} &= -A_{\sigma,m}\frac{\partial \mathcal{L}}{\partial A_{\sigma,4}} - g_{,m}\frac{\partial \mathcal{L}}{\partial g_{,4}} \\ &= -i\left(U A_{n,m}\phi_{,n} + \frac{\partial \mathcal{L}}{\partial \omega}\phi_{,m} + \frac{\partial \mathcal{L}}{\partial \xi}\phi g_{,m}\right) \ . \end{aligned} \tag{A4.4}$$

Substituting (A4.4) and (A4.1) into the definitions (5.32) and (5.33) of the orbital and spin angular momenta L_k and S_k, respectively, one finds, in the static case:

$$(A4.5)\quad L_k = -\epsilon_{klm}\int x_l\left(UA_{n,m}\phi_{,n} + \frac{\partial\mathcal{L}}{\partial\omega}\phi_{,m} + \frac{\partial\mathcal{L}}{\partial\xi}\phi g_{,m}\right)d^3x \quad \text{and}$$

$$(A4.6)\quad S_k = U\,\epsilon_{klm}\int A_l\phi_{,m}\,d^3x \quad , \text{ or,}$$

using the form of the electric field in the static case: $\vec{E} = -\vec{\nabla}\phi$, we get the more familiar expression

$$(A4.7)\quad \vec{S} = U\int \vec{E}\times\vec{A}\,d^3x \quad .$$

Using the field equations one can manipulate the expression for L_k into a form involving S_k. The field equations (A4.3) in the static case are:

$$(A4.8)\quad UA_{n,mn} + VA_{m,nn} + \left(\frac{\partial\mathcal{L}}{\partial\omega}\right)_{,m} - \frac{\partial\mathcal{L}}{\partial\xi}g_{,m} - 2\frac{\partial\mathcal{L}}{\partial\zeta}A_m = 0$$

$$(A4.9)\quad V\phi_{,nn} - 2\frac{\partial\mathcal{L}}{\partial\zeta}\phi = 0 \quad .$$

Subtracting A_m times (A4.9) from ϕ times (A4.8) and integrating gives

$$(A4.10)\quad 0 = \epsilon_{klm}\int x_l\left[\phi\left(UA_{n,mn}+VA_{m,nn}+\left(\frac{\partial\mathcal{L}}{\partial\omega}\right)_{,m} - \frac{\partial\mathcal{L}}{\partial\xi}\phi g_{,m} - VA_m\phi_{,nn}\right)\right]d^3x.$$

Using the expression (A4.5) for L_k, the relation (A4.10) becomes

$$(A4.11)\quad L_k = -\epsilon_{klm}\int x_l\left(U(\phi A_{n,m})_{,n} + \left(\phi\frac{\partial\mathcal{L}}{\partial\omega}\right)_{,m} + V(\phi A_{m,nn} - A_m\phi_{,nn})\right)d^3x \ .$$

Now look at each of the terms in (A4.11) separately. The term multiplied by U can be written as

$$-\epsilon_{klm}\int x_l(\phi A_{n,m})_{,n}\,d^3x = -\epsilon_{klm}\int\left((x_l\phi A_{n,m})_{,n} - (\phi A_l)_{,m} + \phi_{,m}A_l\right)d^3x$$

(A4.12) (continued) $$= -\varepsilon_{klm} \int\!\!\int \left((x_l \phi A_{n,m})_{,n} - (\phi A_l)_{,m} \right) d^3x - \frac{1}{U} S_k ,$$

using the expression (A4.6) for S_k. The middle term in (A4.11) is

(A4.13) $$-\varepsilon_{klm} \int x_l \left(\phi \frac{\partial \mathcal{L}}{\partial \omega} \right)_{,m} d^3x = -\varepsilon_{klm} \int \left(x_l \phi \frac{\partial \mathcal{L}}{\partial \omega} \right)_{,m} d^3x ,$$

because $\varepsilon_{klm} = -\varepsilon_{kml}$. Finally, the term in (A4.11) multiplied by V is

$$-\varepsilon_{klm} \int x_l (\phi A_{m,nn} - A_m \phi_{,nn}) \, d^3x$$

$$= -\varepsilon_{klm} \int \left(\left(x_l (\phi A_{m,n} - A_m \phi_{,n}) \right)_{,n} - (\phi A_m)_{,l} + 2\phi_{,l} A_m \right) d^3x$$

$$= -\varepsilon_{klm} \int\!\!\int \left(\left(x_l (\phi A_{m,n} - A_m \phi_{,n}) \right)_{,n} - (\phi A_m)_{,l} \right) d^3x + \frac{2}{U} S_k ,$$

using (A4.6) for S_k once again.

Substituting (A4.12), (A4.13), and (A4.14) into the expression (A4.11) for L_k gives

(A4.15) $$L_k = \left(\frac{2V}{U} - 1 \right) S_k - \varepsilon_{klm} \int \left(\phi \left(x_l \frac{\partial \mathcal{L}}{\partial \omega} + (V-U) A_l \right) \right)_{,m} d^3x$$

$$- \varepsilon_{klm} \int \left(x_l (U \phi A_{n,m} + V \phi A_{m,n} - V A_m \phi_{,n}) \right)_{,n} d^3x .$$

The integrals appearing explicitly in (A4.15) can all be converted to surface integrals at spatial infinity and will therefore vanish if the fields A_μ and g go to zero fast enough as $r \to \infty$. For example, if, as $r \to \infty$, ϕ goes as $\frac{q}{r}$, $\vec{A}$ goes to zero faster than $\frac{1}{r}$ and $\frac{\partial \mathcal{L}}{\partial \omega}$ goes to zero faster than $\frac{1}{r^2}$, then the integrals vanish. Therefore, for time-independent sufficiently localized solutions, for all Lagrangians of the form (5.39), that is,

$$\mathcal{L} = \frac{U}{2} A_{\sigma,\rho} A_{\rho,\sigma} + \frac{V}{2} A_{\sigma,\rho} A_{\sigma,\rho} + \bar{\mathcal{L}} (A_{\sigma,\sigma} \, ; \, \tfrac{1}{2}(g_{,\sigma})^2 ; \, g_{,\sigma} A_\sigma \, ; \, A_\sigma^2 \, ; \, g),$$

the following relation exists between the spin and orbital angular momenta:

(A4.16) $$L_k = \left(\frac{2V}{U} - 1\right) S_k \quad .$$

Using the explicit expression (A4.7) for S_k we have

$$\vec{S} = \quad U \int \vec{E}\times\vec{A}\, d^3x$$

(A4.17) $$\vec{L} = (2V-U) \int \vec{E}\times\vec{A}\, d^3x$$

$$\vec{J} = \vec{L} + \vec{S} = \quad 2V \int \vec{E}\times\vec{A}\, d^3x \quad ,$$

where U and V are constants, and $\vec{E} = -\vec{\nabla}\phi$, is the electric field.